LECTURE NOTES

ON

DIGITAL SIGNAL PROCESSING

DIPLOMA

Subject code-TH3

6^{TH} SEMESTER, E&TC ENGINEERING

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Chapter-1

1.1 **Signal** → It is a physical quantity which varying with respect to time, temperature, pressure or any other independent variables

or

Signal is a function of one or more independent variables that carries some information is represent a physical phenomena.

Ex-

System → It is a physical device that operates on the signal to change some characteristics of the signal (amplitude, phase, frequency, shape). The example of a system is a filter which is used to reduce the noise or to select the frequency component.

Signal processing → It is an operation that changes the characteristics of a signal such as amplitude, shape, phase, & frequency etc. of a signal.

or Changing the basic nature of signal to obtain the desired shaping of the input signal is called signal processing.

→ Signal processing is concerned with the representation, transformation, and manipulation of signals & the information they contain.

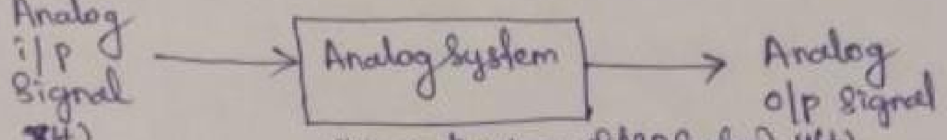
Signal processing is of two types depending upon the type of signal to be processed.

1. Analog Signal Processing (ASP)
2. Digital Signal processing (DSP)

Analog Signal Processing →

In analog signal processing continuous-amplitude continuous-time signals are processed. Various types of analog signals are processed through low pass filters, high pass filters, band pass filters & band reject filters to obtain the desired shaping of the i/p signal.

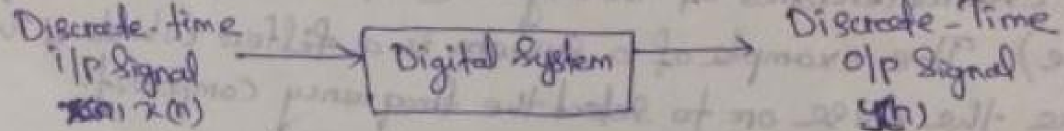
Ex:- modulating audio signal,



(Block diagram of ASP System) y(t)

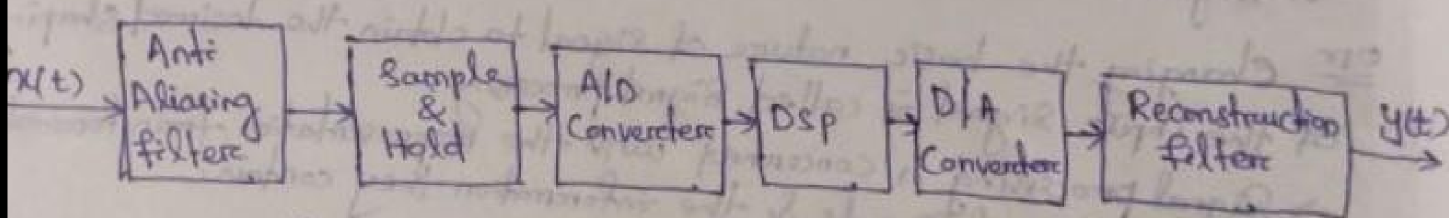
Digital Signal Processing →

Digital signal processing (DSP) is a numerical processing of signals on a digital computer.



(Block diagram of DSP System)

1.1.1 Basic Elements of Digital Signal Processing System →



(Block diagram of DSP System)

1. The i/p signal is applied to the anti-aliasing filter. The low pass filter removes the high frequency noise & to band-limit the signal.
2. The sample & hold provides the discrete time signal to A/D Converter.
3. The ADC converts analog signal to digital signal.
4. The DSP may be a large programmed digital computer programmed to perform the desired operation on the i/p signal.
5. The o/p of DSP is converted to analog signal by DAC.
6. The high frequency components in DAC o/p is ~~reduced~~ rejected by the reconstruction filter.

1.2 Advantages of Digital Signal processing (DSP) over Analog Signal Processing (ASP) →

1. Digital signal processing operations can be changed by changing the program in digital programmable system.
2. DSP is flexible in configuration.
3. There is a better control of accuracy in digital systems compared to analog systems.
4. Digital signals are easily stored on magnetic media such as magnetic tape without loss of quality of reproduction of signal.
5. Digital signals can be processed off line ; i.e, these are easily transported.
6. The DSP can be implemented sophisticated algorithm.
7. Digital ckts are less sensitive to tolerances of ~~computer~~ component values.
8. Digital systems are independent of temperature, ageing & other external parameters.
9. Digital ckts ~~can~~ can be reproduced easily in large quantities at comparatively lower ~~co~~ cost.
10. Cost of processing per signal in DSP is reduced by time-sharing of given processor among a no. of signals.
11. Digital system can be cascaded without any loading problems.

Limitations of DSP →

1. System complexity :→ System complexity increased in the digital processing of an analog signal because of the devices such as A/D & D/A converters.
2. Bandwidth limited by sampling rate :→ Band limited signals can be sampled without information loss if the sampling rate is more than twice the bandwidth.
3. Power consumption :→ A variety of analog processing algorithms can be implemented using passive ckt employing inductors, capacitors & resistors that don't need any power, whereas a DSP chip containing over 4 lakh transistors dissipates more power.

Applications of DSP

1. Transmission line
2. Advanced optical fibre communication
3. Analysis of Sound & Vibration Signals
4. Implementation of Speech recognition algorithm
5. VLSI technology
6. Telecommunication Networks
7. Microprocessor Systems
8. Satellite Communication
9. Astronomy
10. Consumer Electronics
11. Image Processing
12. Military
13. Seismology

1.2. Classify Signals →

A Signal can be anything which conveys information. For example → a picture of a person gives you information regarding whether he is short or tall, fair or black etc.

→ Mathematically signal is defined as a function of one or more depend variables that conveys information about the state of a system. For example :-

a) A speech signal is a function of time. Here independent variables is time & dependent variable is amplitude of speech signal.

b) A picture containing varying brightness is a function of two spatial variables. Here independent variables are spatial coordinate(X, Y) & dependent variable is brightness or amplitude of picture.

Classification of Signals →

Signals can be classified based on parameters used to classify them such as

a) Nature of independent variable such as time
- → Continuous time signals
- → Discrete time signal

b) Nature of dependent variable
- → Analog signal
- → Digital signal

c) No. of independent variables
- → One dimensional signal
- → Two dimensional signal
- → Multidimensional signal

d) Based on nature of indeterminancy
- → Deterministic signal
- → Random signal

c) Based on causality
- → Causal signal
- → Anti causal signal
- → Non causal signal.

2 Continuous-time Signals →

Continuous time signals are the signals that are defined over a continuous range of time. i.e, time can assume any value from $(-\infty, \infty)$.

→ For every instance of time there exists ~~exis~~ a unique & ~~signa~~ single value of function $f(t)$.

→ These signals are also called as analog signals.

~~Ex → Seismic Signals, Speech Signal~~

~~Discrete time Signals~~ :→

Ex:→ Mathematical function ($A \sin \Omega t$)

Ex→ Waveform of AC power supply

Discrete time Signals :→

The discrete time signals are defined at discrete instant of time & denoted by $x(n)$

Ex→ Business, the runs scored by a team in each over in a one-day international cricket match.

* In discrete time signals, we are just ignoring the unwanted information in the signal by taking the ~~am~~ signal amplitudes at discrete instant of time.

One dimension signal

If the signal is a function of only one independent variable, such signal is ~~call~~ referred to as one dimensional signal.

Ex:→ A noisy voice signal is a one dimensional signal is a function of only time.

Two dimensional Signal

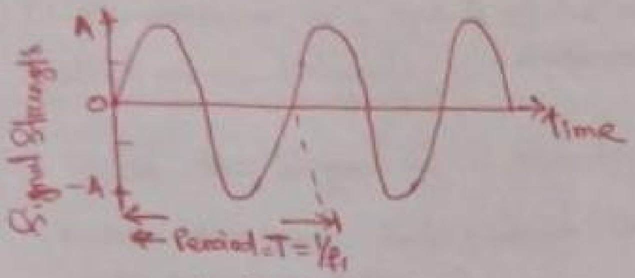
If the signal is a function of two dependent variables, then it is referred to as two dimensional signal.

Ex:→ Simple black & white picture is a function of intensity in the figure is a two-dimensional signal is a function of spatial coordinates X & Y. At each point (X,Y) an intensity value is assigned & mapped onto computer screen as a 2D image.

Multidimensional Signal

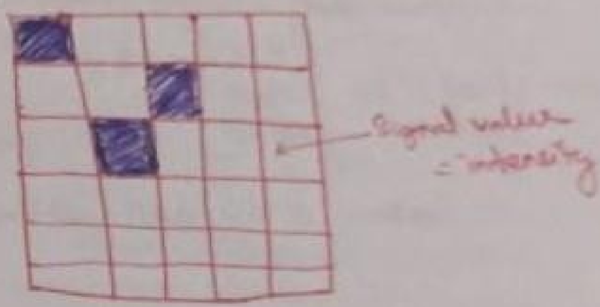
Multidimensional signal is a function of more than two variables.

Ex:→ A video signal is a ~~fun~~ function of three independent variables which are time & two spatial coordinates (X,Y).



a) Sine wave

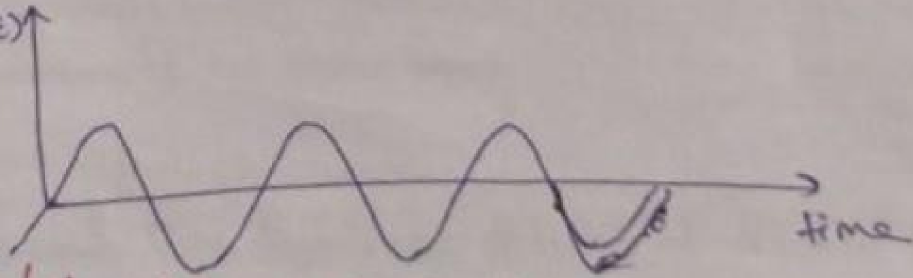
– 1D Signal



– 2D Signal

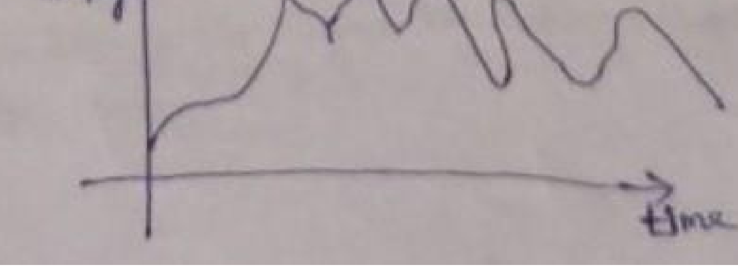
Deterministic Signal

Deterministic signals are those signals whose values are completely specified for any given time. Thus a deterministic signal can be modeled by a known function of time.



Random Signals / Non-deterministic Signals :–

Random signals are ~~also~~ ~~called~~ those signals that take random values at any given time & ~~~~ are also called non-deterministic signals.



(4.) **Multichannel Signals :->**

The signals which are generated from a multiple sources are called Multichannel Signals

Ex:-> ECG waveform -> To generate ECG waveform, different leads are connected to the body of a patient. Each lead is acting as an individual channel. Since there are n number of leads, the final ECG waveform is a result of the multichannel signal. Mathematically expressed as

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$ if 3 leads are used.

Multidimensional Signals :->

If the signal is a function of more than one independent variables then it is called multidimensional signal.

Ex:-> Speech signal, Seismic Signal

2.2 **Continuous time Signal :->**

Continuous time signals are the signals that are defined over a continuous range of time i.e. time can assume any values from $(-\infty, \infty)$

-> These signals are also called as analog signals

Ex:- Mathematical function (A Sin 2t)

Ex -> Waveform of AC Power Supply

Discrete time Signals :->

The discrete time signals are defined at discrete instant of time & denoted by $x(n)$

Ex => Business, the runs scored by a team in each over in a one day international cricket match.

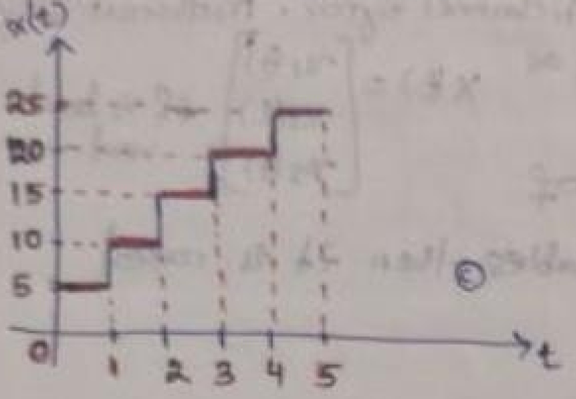
-> In discrete time signals, we are just ignoring the unwanted information in the signal by taking the signal amplitudes at discrete instant of time.

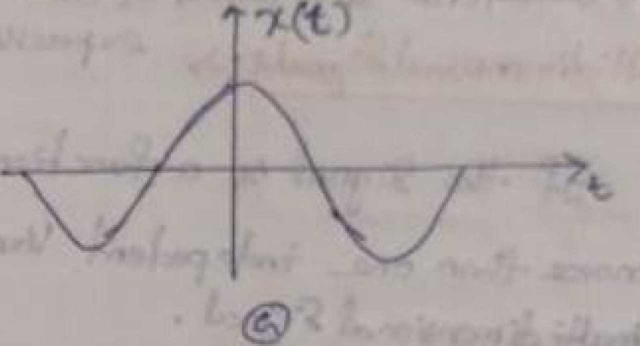
1.2.3 Continuous Valued Signals :-

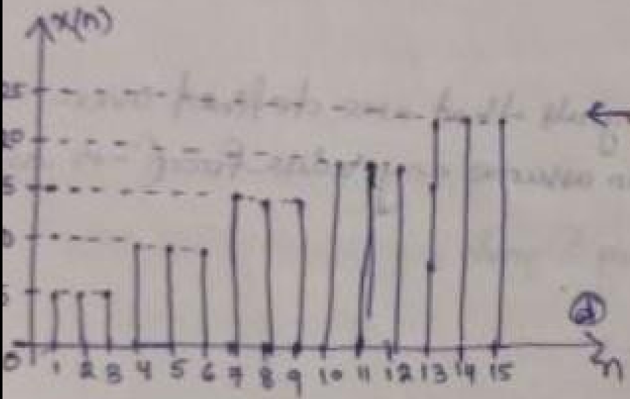
If the Variation in the amplitude of signals is continuous or discrete in nature.

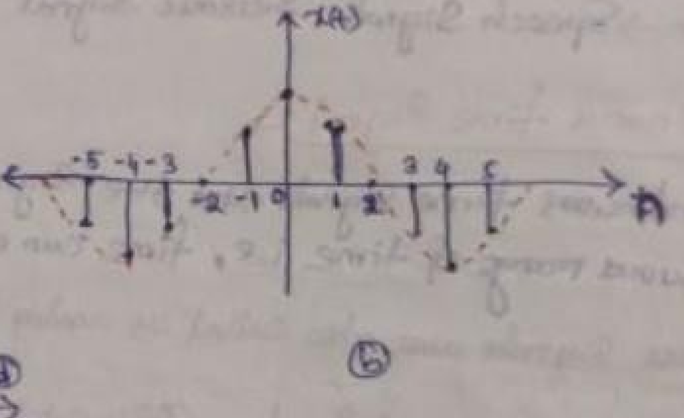
Discrete Valued Signals :-

If the Variation in the amplitude of signals is not continuous, but the signal has certain discrete amplitude levels then such signals is called as discrete valued signals.









(Discrete valued Signals) (Continuous-Valued Signals)

1.3 Discuss the concept of frequency in Continuous time & discrete time Signals.

$$\text{Frequency}(f) = \frac{1}{\text{Time Period}}$$

$$f = \frac{1}{T}$$

Definition (Freq) → It is the no. of cycles per second.

1.2.1 Continuous time Sinusoidal Signals :→

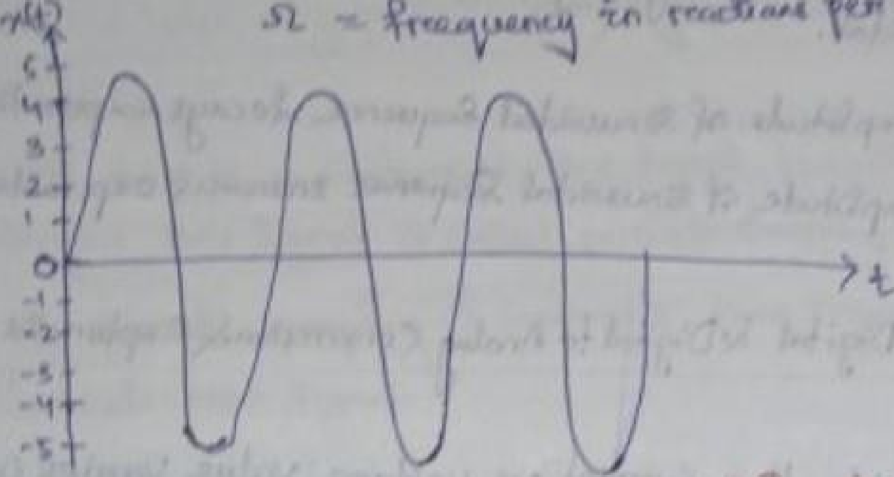
A continuous time Signal Sinusoidal Signal is given by

$$x(t) = A\sin(\Omega t + \theta)$$

Where A = Amplitude

θ = phase angle in radians

Ω = frequency in radians per second



(Continuous time Sinusoidal Signals)

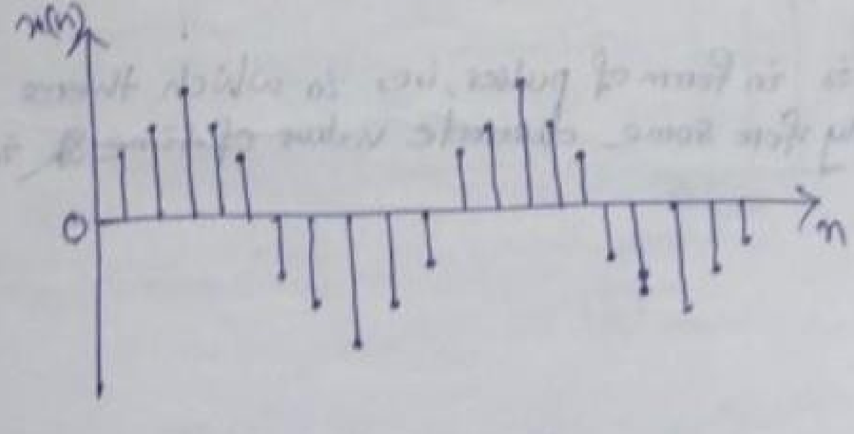
Discrete time Sinusoidal Signals :→

The discrete time Sinusoidal Signal is given by

$$x(n) = A\cos(\omega_0 n + \phi)$$

where ω_0 = frequency (radians per Sample)

ϕ = phase in radians



1.3.5 Harmonically related complex exponential Signal :→

The discrete-time exponential Signal is given by

$$x(n) = a^n e^{j(\omega_0 n + \phi)}$$

$$= a^n \cos(\omega_0 n + \phi) + j a^n \sin(\omega_0 n + \phi)$$

→ For $|a| = 1$, the real & imaginary parts of complex exponential Sequence are sinusoidal.

→ For $|a| < 1$, the amplitude of sinusoidal Sequence decays exponentially

→ For $|a| > 1$, the amplitude of sinusoidal Sequence increases exponentially.

1.4 Discuss Analog to Digital & Digital to Analog Conversion & Explain the following

Analog Signal :→

A Signal in which the current or voltage value varies continuously with time is called the analog signal.

The most simplest form of analog signal is the sinusoidal signal.

Ex :→ the signal obtained from the sounds of speech, music, vibrating tuning fork etc.

→ the analog signals which are obtained by microphone as voltage & current signal varying with time.

Digital Signal :→

A Signal which is in form of pulses, i.e, in which there is voltage or current only for some discrete values of time & is called digital signal.

1.4) Sampling of Analog Signal:-

The process of converting a continuous time signal to discrete-time signal is known as **Sampling**.

→ There are many ways to sample a continuous-time signal. Here we have only periodic sampling. It is also called Uniform sampling.

→ If $S_a(t)$ is a continuous-time signal. Periodical measurement of continuous-time signal is called periodic sampling or Uniform Sampling.

→ By periodic sampling of continuous-time signal, we can get discrete-time signal.

$$\text{Discrete-time Signal } S_a(nT_s) = S_a(t)\big|_{t=nT_s}$$

Where T is the sampling period & reciprocal of sampling is termed as sampling frequency F_s i.e., $\boxed{F_s = \frac{1}{T_s}}$

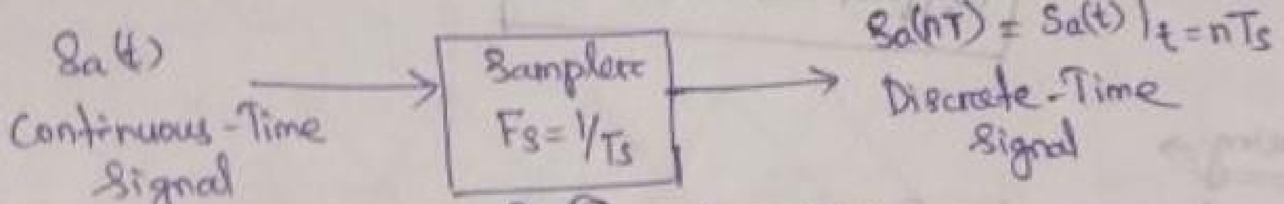
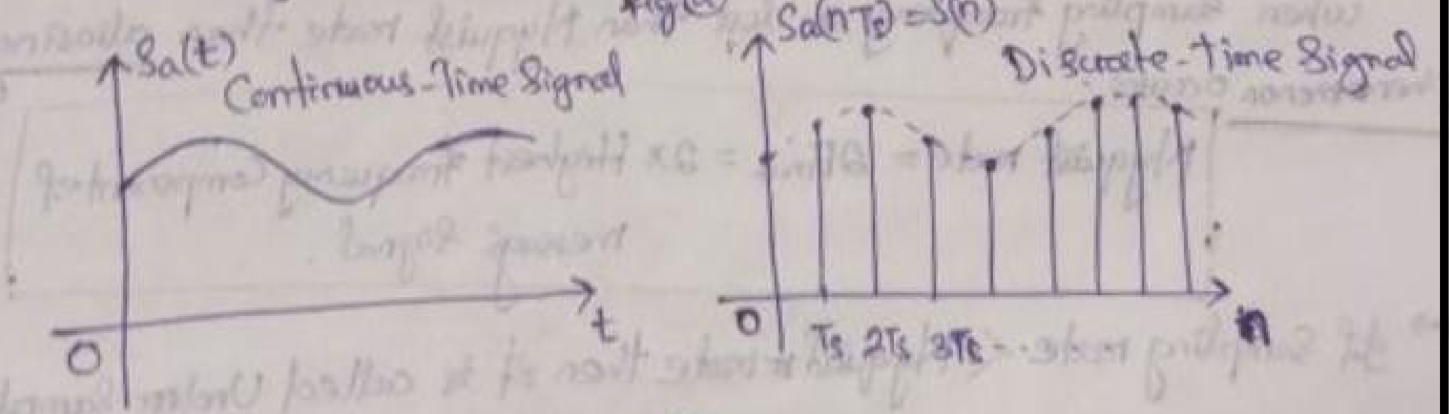


Fig (a)



(b)

Fig:→ (a) Block diagram of a sampler (b) Periodic sampling of Continuous-time signal.

1.1.2 Sampling theorem :→

It is stated as 'for perfect reconstruction of sampled signal at receiver end, sampling rate or sampling frequency should be greater or equal to the maximum or highest frequency present in the signal.

According to the sampling theorem

Sampling rate $\geq 2F_{max}$

$$\boxed{F_s \geq 2F_m}$$

Nyquist Rate :→

Nyquist rate is defined as minimum sampling rate required for perfect reconstruction of sampled signal at receiver end.

If any signal has highest frequency component Fmax, then.

$$\boxed{\text{Nyquist rate} = 2 \times F_{max}}$$

Aliasing :→

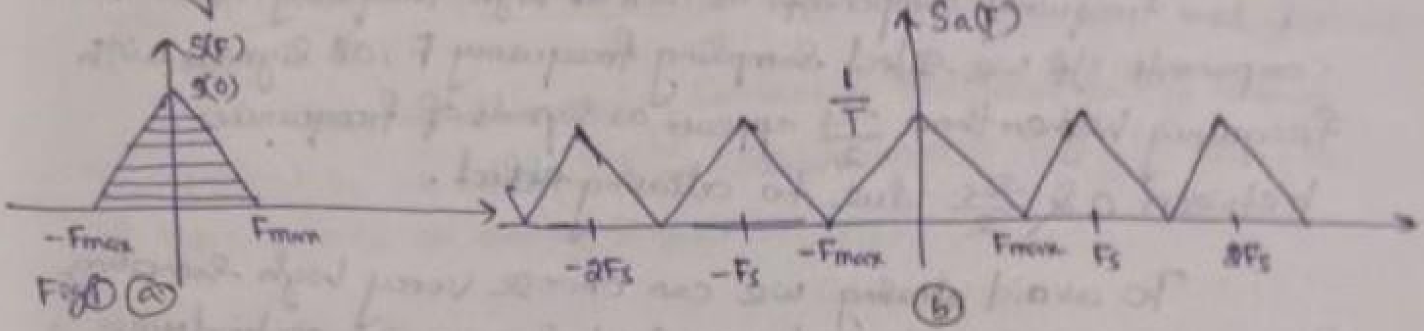
when sampling frequency is less than Nyquist rate than aliasing phenomenon occurs.

$$\boxed{\text{Nyquist rate} = 2F_{max} = 2 \times \text{Highest frequency Component of message signal.}}$$

→ If sampling rate < Nyquist rate then it is called Under Sampling & in this aliasing effect occurs

→ If sampling rate > Nyquist rate then it is called over sampling (Up Sampling). Infact this is a suitable & necessary condition for sampling process.

→ Aliasing effect is defined as a phenomenon of high frequency component in a spectrum of a signal seemingly taking on the identity of a lower frequency in the spectrum of its sampled version.



(a) Spectrum of a band-limited analog signal (b) Spectrum of a sampled version signal S(t) for a sampling frequency $F_s = 2F_{max}$

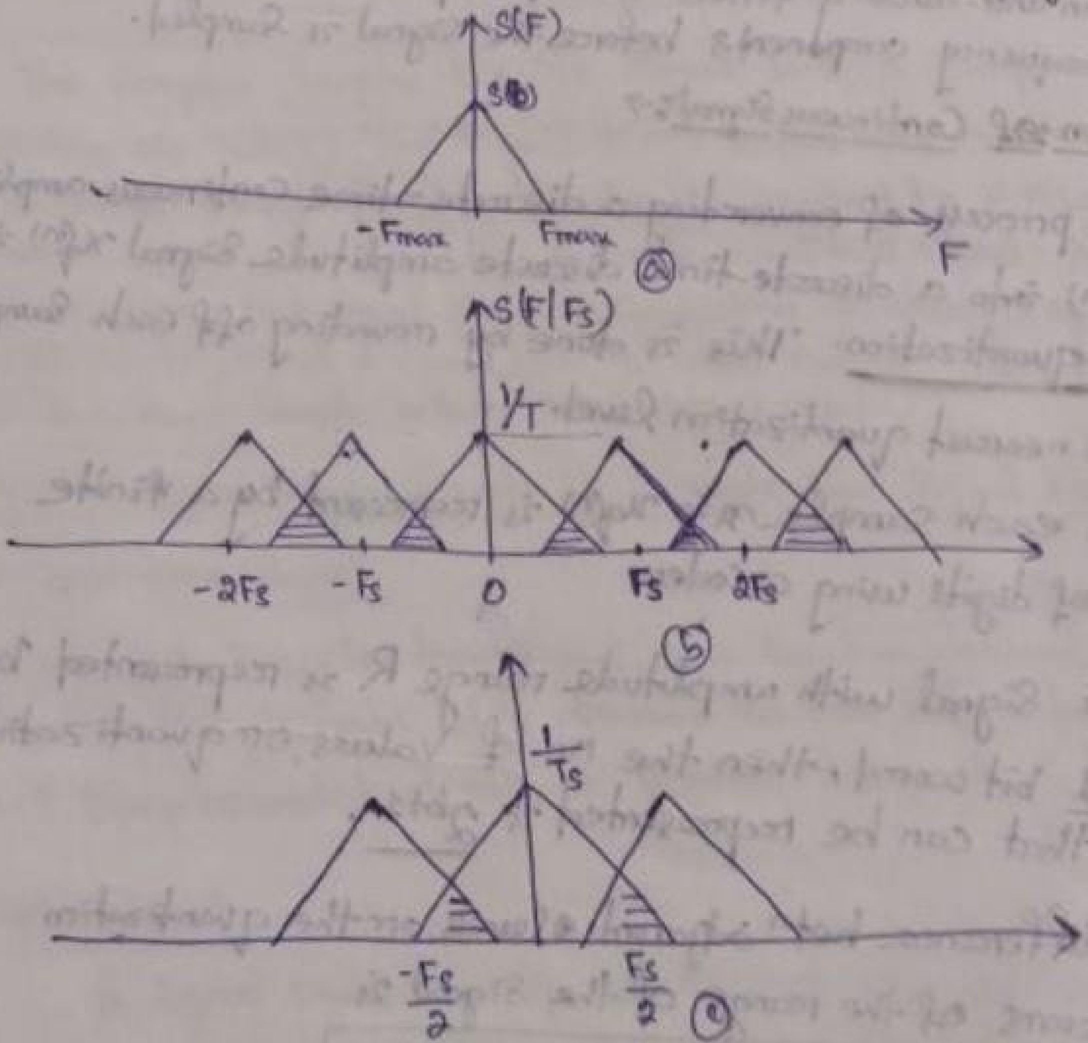


Fig 1.1 a

Fig 1.1: The effect of under sampling an analog signal on its digital frequency response showing aliasing around the folding frequency $F_s/2$

Anti-Aliasing filter:-

In practice, communication signals have frequency spectra consists of low frequency components as well as high-frequency noise components. If we select sampling frequency F, all signals with frequency higher than $\frac{\Omega_s}{2}$ appear as signals of frequencies between 0 & $\frac{\Omega_s}{2}$ due to aliasing effect.

To avoid aliasing we can choose very high sampling frequency. But sampling at very high frequencies introduces numerical errors. Therefore, to avoid aliasing errors caused by the undesired high frequency signals, an analog lowpass filter, called an anti-aliasing filter is used prior to sampler to filter high frequency components before the signal is sampled.

1.4.3 Quantization of Continuous Signal:→

The process of converting a discrete-time continuous-amplitude signal $x(n)$ into a discrete-time discrete amplitude signal $x_q(n)$ is known as quantization. This is done by rounding off each sample in $x(n)$ to nearest quantization level.

→ Then each sample in $x_q(n)$ is represent by a finite number of digits using a coder.

→ If a signal with amplitude range R is represented by an b+1 bit word, then the no. of values, or quantization levels, that can be represented is 2^{b+1}.

→ The difference betn adjacent levels, or the quantization steps in terms of the range of the signal is

$$\boxed{q = \frac{\text{range of signal}}{\text{No. of quantization levels}} = \frac{R}{2^{b+1}}}$$

1.4.5 Digital to Analog Conversion

Digital signal $x(n)$ → [DAC] → [interpolator] $\xrightarrow{x_a(t)}$ [Low pass filter smoothing] → Analog o/p signal $x(t)$

Here the digital to analog Converter produces an o/p voltage is applied to an interpolator which Converts the o/p samples of DAC into an analog signal.

Analog to Digital Conversion

$x_a(t)$ → [Sampler] $\xrightarrow{x_a(nT)}$ [Quantizer] $\xrightarrow{x_q(n)}$ [Coder] → Digital Numbers

The Sampler samples the i/p signal with a sampling interval T, producing o/p $x_a(nT)$. The signal $x(nT)$ is discrete-in-time but continuous in amplitude. The o/p of the sampler is applied to a quantizer. It Converts $x(nT)$ into discrete, discrete-amplitude signal. After sampling & quantization, the final step is converting an analog signal to a form acceptable to digital computer is called coding. The coder maps each quantized sample value in digital word.

1.4.6 Analysis of Digital System Signals Vs. Discrete time Signal Systems :→

Discrete time Signal

A Discrete-time signal is a function defined only at a particular time instants. It is discrete in time but continuous in amplitude.

Ex → temp recorded at regular interval of time in a day.

Digital Signal

A digital signal is a special form of discrete time signal which is discrete in both time & amplitude obtained by quantizing each value of the discrete-time signal. These signals are called digital because their samples are represented by numbers or digits.

Ex - the o/p from a digital computer.

Problems

Question 1 An ~~ah~~ analog signal is represented by

$x_a(t) = \sin(480\pi t) + 3\sin(720\pi t)$.

What is the Nyquist rate of the signal.

Soln

$\omega_1 = 480\pi$

$2\pi f_1 = 480\pi$

$\Rightarrow 2f_1 = 480$

$\Rightarrow f_1 = \frac{480}{2} = 240\,Hz$

Similarly $f_2 = \frac{720}{2}\,Hz = 360\,Hz$

Nyquist rate $F_{s(min)} = 2F_{max}$

Here $F_{max} = 360\,Hz = f_2$

So, $F_{s(min)} = 2 \times 360 = 720\,Hz$ Ans

Signal Processing → It is an operation that changes the characteristics of a signal such as amplitude, shape, phase & frequency of a signal.

2. Discrete Time Signals & Systems

2.1. State & explain discrete time signal.

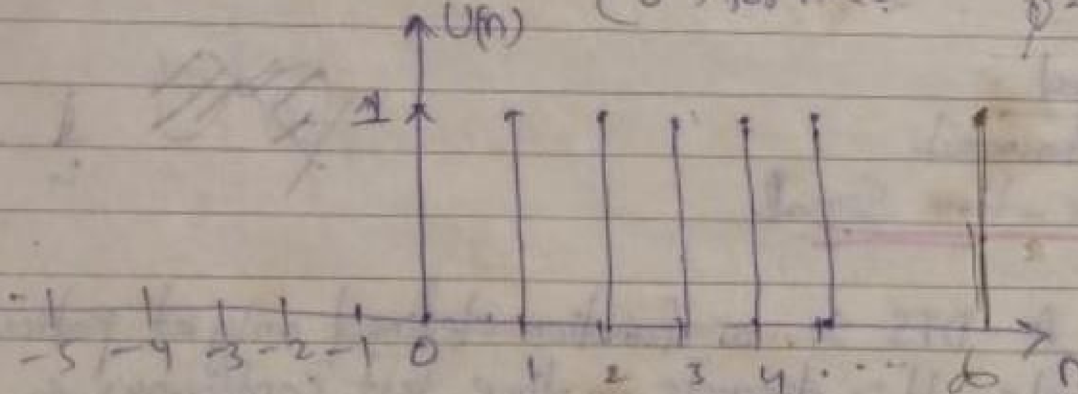
2.1.1. Discuss some elementary discrete time signal.

Unit Step Sequence

The unit step sequence is defined as

$$U(n) = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

$0 < n < \infty$

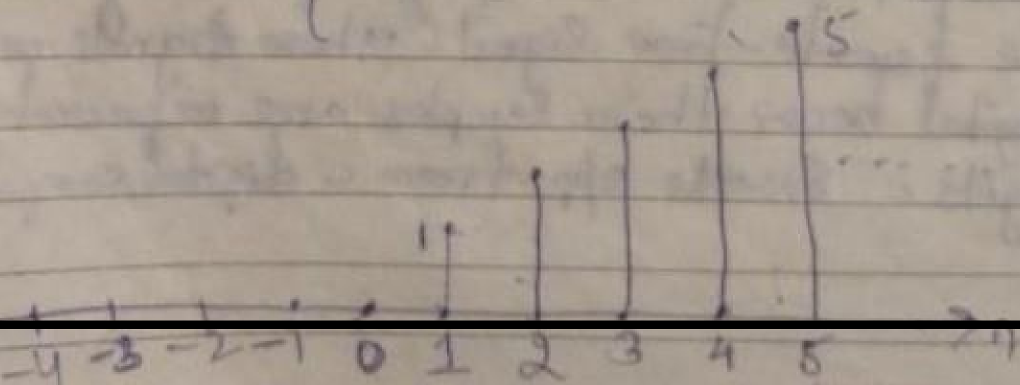


Unit ramp Sequence

The unit ramp sequence is defined as

$$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

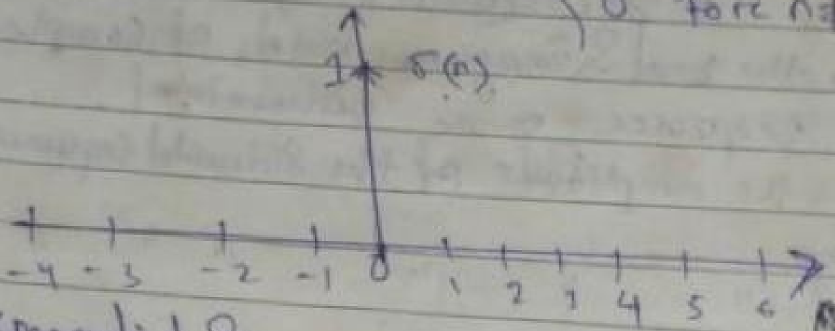
$n = 0$

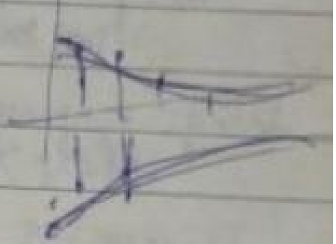


Unit Sample Sequence

The unit sample sequence (unit impulse sequence) is defined as

$$\delta(n) = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n\neq 0 \end{cases}$$

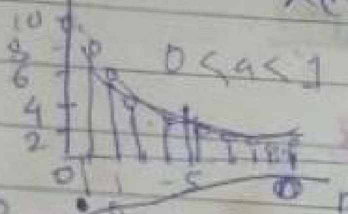




Exponential Sequence

The exponential sequence is a sequence given by

$$x(n) = a^n \text{ for all } n$$



$e^{j\theta} = \cos\theta + i\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

Sinusoidal Signal

A discrete-time Sinusoidal Signal is given by

$$x(n) = A\cos(\omega_0 n + \phi)$$

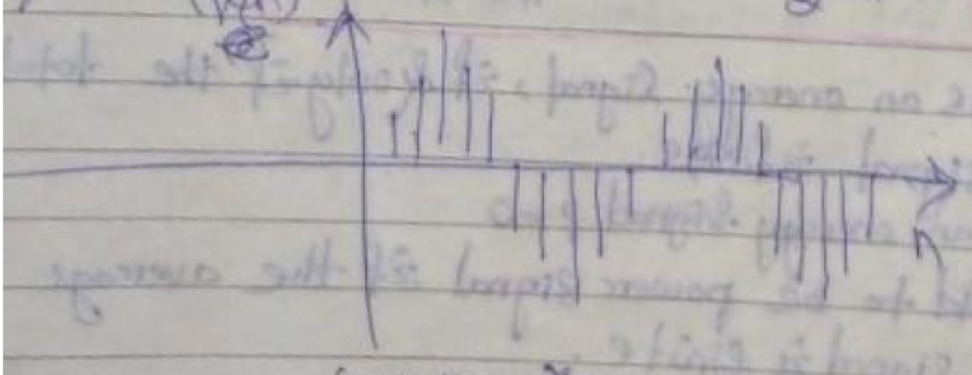
where ω_0 = frequency (in radians per Sample)

ϕ = phase in radians

$\cos\theta + i\sin\theta$

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

$j(\phi + \omega_0 n)$



$(\cos\phi + i\sin\phi)(\cos\omega_0 n + i\sin\omega_0 n)$

$= \cos(-\phi) + i\sin(-\phi) + \cos(\omega_0 n) + i\sin(\omega_0 n)$

$= \cos\phi$

Complex exponential Signal

The discrete-time complex exponential signal is given by

$$x(n) = a^n e^{j(\omega_0 n + \phi)}$$

$$= a^n \cos(\omega_0 n + \phi) + j a^n \sin(\omega_0 n + \phi)$$

$|a| = 1$, the real & imaginary parts of complex exponential sequence are sinusoidal.

$|a| < 1$, the amplitude of the sinusoidal sequence decays exponentially.

$|a| > 1$, " " " " " " increases exponentially.

2. classify discrete time signals.

Classification of discrete time signal

Energy power & power signals

Energy signals

For a discrete-time signal $x(n)$, the energy E is defined as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Power signals

The average power of a discrete-time signal $x(n)$ is defined as

$$\boxed{P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2}$$

* A signal is an energy signal, if & only if the total energy of the signal is finite.

For an energy signal $P = 0$

* A signal is said to be power signal if the average power of the signal is finite.

For a power signal $E = \infty$

Q11 Determine the values of power & energy of the following signals

i) $x(n) = \left(\frac{1}{3}\right)^n u(n)$ ii) $x(n) = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}$

iii) $x(n) = \sin\left(\frac{\pi}{4}n\right)$ iv) $x(n) = e^{2n} u(n)$

Ans i) $x(n) = \left(\frac{1}{3}\right)^n u(n)$

$\because u(n) = 1, n \geq 0$
$= 0, n < 0$

$1 + a + a^2 + \cdots = \frac{1}{1-a}$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} \left[\left(\frac{1}{3}\right)^n\right]^2$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n = \left(\frac{1}{9}\right)^0 + 1 + \frac{1}{9} + \frac{1}{9^2} + \cdots$$

$$= \frac{1}{1 - \frac{1}{9}} = \frac{9}{8}$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{9}\right)^n$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \left[\frac{1 - \left(\frac{1}{9}\right)^{N+1}}{1 - \frac{1}{9}}\right]$$

$$= 0$$

E is finite & P is zero. Therefore the signal is an energy signal.

ii) $x(n) = e^{j\left(\frac{\pi}{2}n + \pi/4\right)}$ $\because e^{j(\omega + \theta)} = 1$

$$E = \sum_{n=-\infty}^{\infty} \left| e^{j\left(\frac{\pi}{2}n + \pi/4\right)} \right|^2$$

$e^{j\theta} = \cos\theta + i\sin\theta$
$= \cos^2\theta + \cdots$

$$= \sum_{n=-\infty}^{\infty} 1 = \infty$$

$\frac{1}{9}^0 + \frac{1}{9} + \frac{1}{9^2} + \cdots$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)} \right|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1 \qquad \boxed{\sum_{n=-N}^{N} 1 = 2N+1}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}(2N+1) = 1$$

i> $x(n) = \sin\left(\frac{\pi}{4}n\right)$

$$E = \sum_{n=-\infty}^{\infty} \left|\sin^2\left(\frac{\pi}{4}n\right)\right|$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1-\cos\left(\frac{\pi}{2}n\right)}{2}\right]$$

$$= \infty.$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|\sin^2\left(\frac{\pi}{4}n\right)\right|$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1-\cos\frac{\pi}{2}n}{2}$$

$$= \frac{1}{2} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= \frac{1}{2}.$$

E is ∞ & P is finite.

$$\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$$

iv) $x(n) = e^{2n} u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=0}^{\infty} e^{4n} = 1 + e^4 + e^8 + \cdots + \infty = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} e^{4n}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{e^{4(N+1)}-1}{e^4-1}\right]$$

$$= \infty$$

The signal is neither power nor energy.

Q11 i) $\cos(\omega_0 n)\, u(n)$ ii) $u(n+2) - u(n-2)$

i)

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} (\cos(\omega_0 n))^2 = \sum_{n=0}^{\infty} \cos^2(\omega_0 n)$$

$$= 1 + \cos^2\omega_0 + \cos^2 2\omega_0 + \cdots + \infty$$

$$= \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} [\cos(\omega_0 n)]^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} (2N+1)$$

$$= 1$$

$E = \infty$, & P is finite

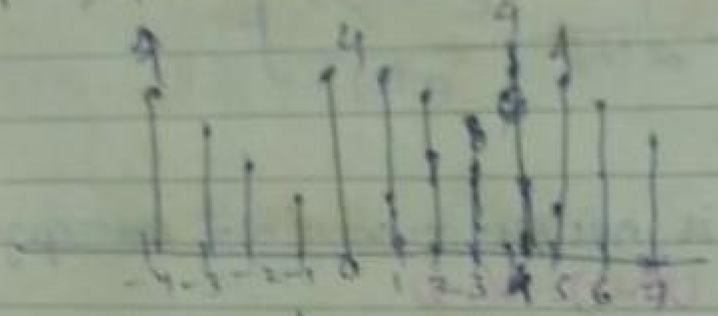
→ Power signal

2. Periodic & Aperiodic Signals

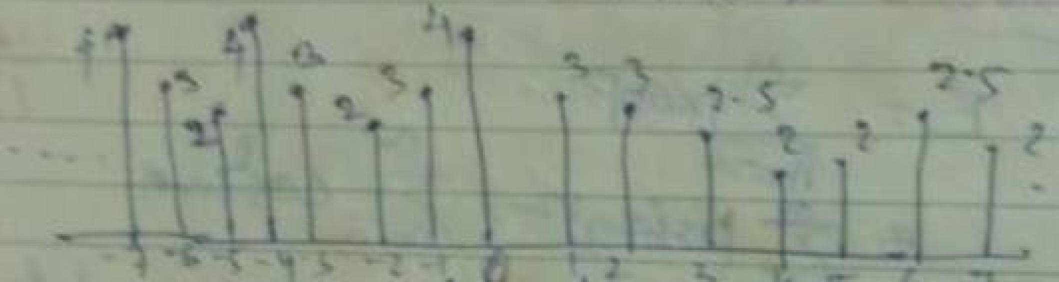
A discrete-time signal $x(n)$ is said to be periodic with period N if & only if

$$x(N+n) = x(n) \text{ for all } n \quad -(1)$$

→ If Eqn doesn't satisfy even for one value of n, then the discrete-time signal is aperiodic.



Periodic



Aperiodic

Properties of Discrete-time Sinusoidal Signal:-

1. The discrete time sinusoids whose frequencies are separated by an integer multiple of 2π are identical

$$x_1(n) = A\sin(\omega_0 n+\theta)$$

$$x_2(n) = A\sin((\omega_0+2k\pi)n+\theta)$$
$$= A\sin(\omega_0 n+\theta)$$
$$= x_1(n).$$

2. The freq. of oscillation of discrete time sinusoidal sequence increases as ω increases from 0 to π. If ω increases from π to 2π then freq. of oscillation decreases

Symmetric (even) & Antisymmetric (odd) Signals

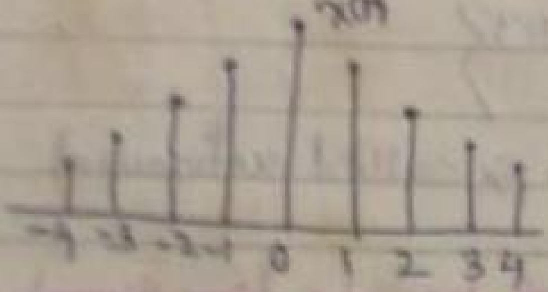
→ A discrete time signal $x(n)$ is said to be a symmetric (even) signal if it satisfies the condition

$$x(-n) = x(n) \quad \forall n$$

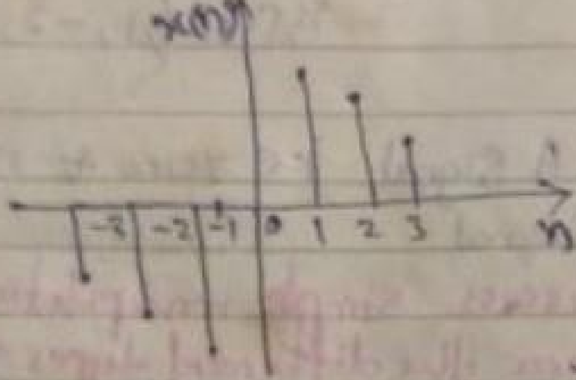
Ex → $x(n) = \cos \omega n$

→ A signal is said to be an odd signal if it satisfies the condition $x(-n) = -x(n) \; \forall n$.

Ex: $A \sin \omega n$



Symmetric (even) (even signal)



(Antisymmetric (odd signal))

$$x(n) = x_e(n) + x_o(n) \quad \text{—(I)}$$

$$x(-n) = x_e(-n) + x_o(-n) \quad \text{—(II)}$$

$$= x_e(n) - x_o(n) \quad \text{—(II)}$$

(I) ~~$x(n) = [x_e(n) - x(n)]$~~

$$2x_e(n) = [x(n) + x(-n)]$$

$$\boxed{x_e(n) = \frac{1}{2}[x(n) + x(-n)]}$$

$$\boxed{x_o(n) = \frac{1}{2}[x(n) - x(-n)]}$$

$$\{\underset{\uparrow}{1}, 2, 3, 4\}$$

~~$n = 0$~~, $n_0 = 0$

$X(0+1) = x(1)$

$X(0+2) = \cancel{x}(2)$

4. Causal & Non causal Signals

→ A signal $x(n)$ is said to be causal if its value is zero for $n<0$ otherwise the signal is non causal.

Ex → $x_1(n) = a^n u(n)$, $x_2(n) = \{\underset{\uparrow}{1}, 2, -3, -1, 2\}$ } Causal

Noncausal

$x_1(n) = a^n u(-n+1)$

$x_2(n) = \{1, -2, \underset{\uparrow}{3}, 4, 3\}$

⇒ A signal i.e zero $\forall\ n \geq 0$ is called anticausal signal

Q13) Discuss simple manipulation of discrete time signal.
What are the different types of operation performed on discrete time signals

i> shifting ii> Time reversal

iii> Time scaling iv> Scalar Multiplication

v> Substraction signals vi> Multiplication

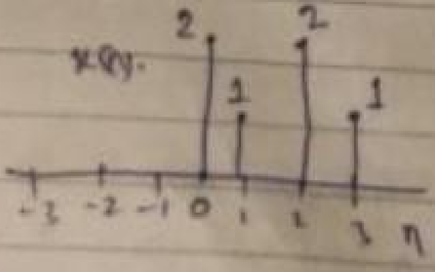
vii> Addition of sequence.

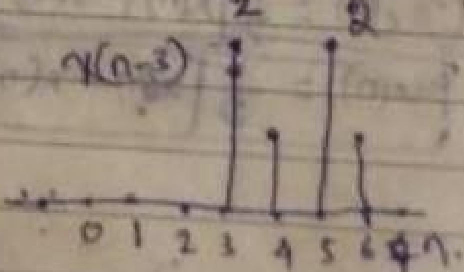
Shifting

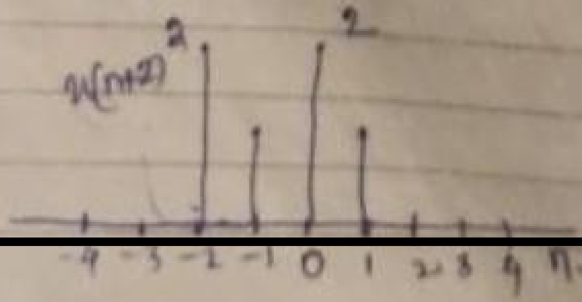
$y(n) = x(n-k)$

$x(n) = i/p$, $y(n) = o/p$.

If $k = +ve$, the shifting delays the sequence
$k = -ve$, the shifting advances the sequence

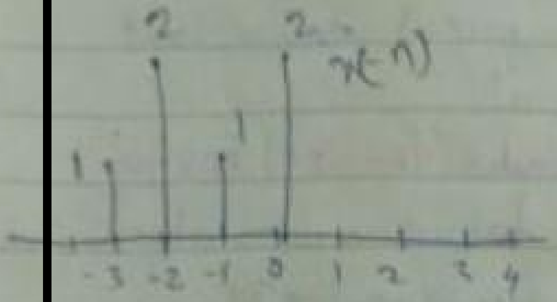


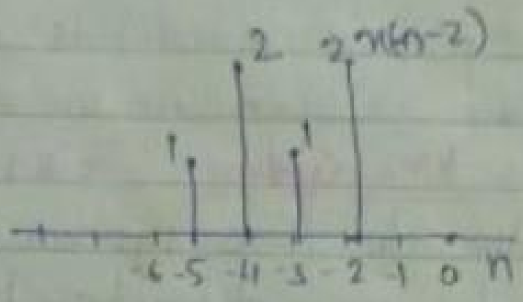


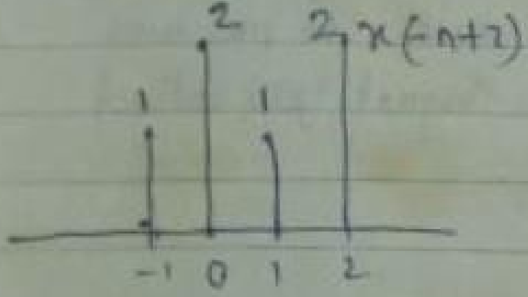


Time reversal

$$x(n) = x(-n)$$







Time Scaling

$$y(n) = x(\lambda n) \quad \lambda = \cancel{\pm 1, \pm 2, 3}\ 0, 1, 2, 3 \ldots$$

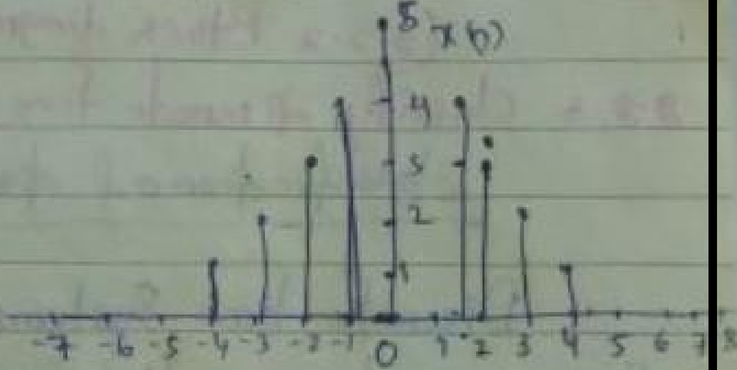
$n = -1,\ y(-1) = x(-2) = 3$

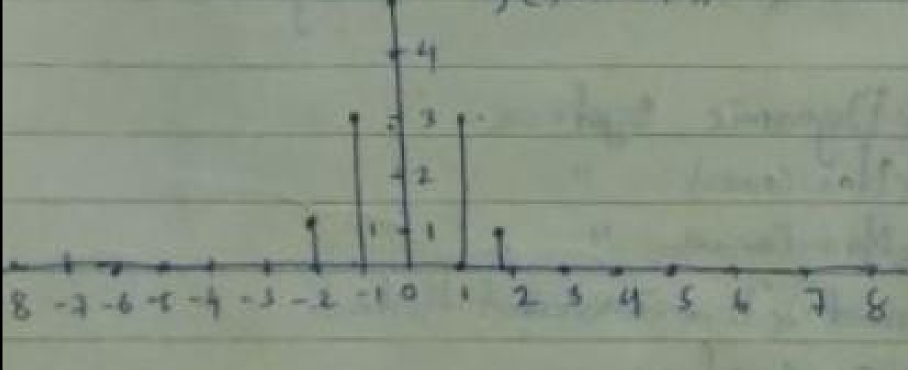
e.g. $y(n) = x(2n)$

$y(0) = x(0) = 5$

$y(1) = x(2) = 3$

$y(2) = x(4) = 1$





2.2 Discuss discrete time system :-

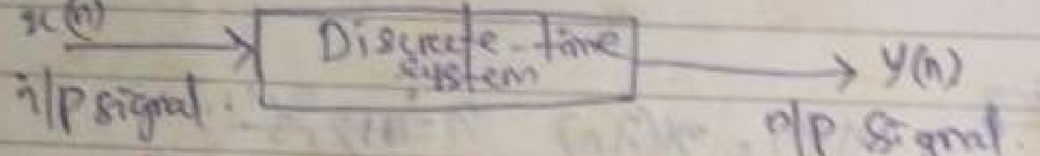
System

A system is defined as a physical device that performs an operations on a signal.

Discrete time system : (2.2.1 Describe i/p - o/p of system)

A discrete time system is a device or an algorithm that operates on a discrete-time i/p signal $x(n)$, according to some well defined rule, to produce another ~~sequence~~ discrete time signal $y(n)$ called the o/p signal.

$$y(n) = T[x(n)]$$



(2.2.2 Block diagram of discrete time system)

2.2.3 Classify discrete time system

: classification of discrete-time system

Discrete-time systems are classified according to their general properties & characteristics. They are -

1. Static & Dynamic system
2. Causal & Non-causal "
3. Linear & Non-linear "
4. Time-Variant & Time-invariant system
5. FIR & IIR system
6. Stable & unstable systems

Static & Dynamic System

→ A discrete time system is called static or memoryless if its o/p at any instant n depends on the i/p samples at the same time, but not on past or future samples of the i/p

→ The system described by

Ex :- $y(n) = a\,x(n)$

$y(n) = a x^2(n)$.

→ A disc. system is said to be dynamic or to have memory if its o/p at any instant n depends on the

→ A dynamic or a system with memory is one in which the past i/ps or o/ps are stored to calculate the present o/p.

$$EX : y(n) = x(n) + 3x(n-1)$$

Q// i> $y(n) = x(n) \cdot x(n-1)$

ii> $y(n) = x^2(n) + x(n)$

Ans i> $y(n) = x(n)\,x(n-1)$

The o/p $y(n)$ depends on the past i/p. the system is dynamic.

ii> $y(n) = x^2(n) + x(n)$

The o/p $y(n)$ depends on the i/p at that instant only. ∴ the system is static.

Ex i> $y(n) = x(2n)$ ii> $y(n) = x^2(n)$.

i> ~~Dynamic~~ Static (memoryless)

ii> static

ii> Causal & Non Causal Systems

* Causal Systems

A system is said to be causal if the o/p of the system at any time n depends only at present & past i/ps, but doesn't depend on future i/ps.

This can be represented mathematically as

$$y(n) = F[x(n), x(n-1), \ldots x(n-2) \ldots]$$

Non Causal System (anticipatory)

A system is said to be causal if the o/p of the system ~~at~~ depends on future i/ps ~~of~~.

Ex:- $y(n) = x(n) + x(n-1) \rightarrow$ causal

$y(n) = x(n) \rightarrow$ Non Causal.

i> $y(n) = x(n) + \frac{1}{x(n-1)}$

ii> $y(n) = x(n^2)$

Ans

i> $y(n) = x(n) + \frac{1}{x(n-1)}$

$n=-1$, $y(-1) = x(-1) + \frac{1}{x(-2)}$

$n=0$, $y(0) = x(0) + \frac{1}{x(-1)}$

$n=1$, $y(1) = x(1) + \frac{1}{x(0)}$

~~for~~ for all the values of n, the o/p depends on ~~past~~ present & past i/ps. therefore the system is causal i> $y(n) = x(n^2)$

$n=-1$, $y(-1) = x(1)$
$n=0$, $y(0) = x(0)$
$n=1$, $y(1) = x(1)$
$n=2$, $y(2) = y(4)$

except for $n=0$ & $n=1$, the system depends on future i/ps. So, the system is non-causal.

ii> Linear & Non-Linear Systems :-

Linear System:

A system that satisfies the superposition principle is said to be a linear system.

Superposition principle states that, the response of the system to a weighted sum of signal should be equal to the corresponding weighted sum of the o/ps of the system to each of the individual i/p signals.

A system is linear if and only if

$$T[a_1x_1(n) + a_2x_2(n)] = a_1T[x_1(n)] + a_2T[x_2(n)]$$

for any arbitrary constants a_1 & a_2.

Non-linear System

→ A relaxed system that doesn't satisfy the superposition principle is called Non-linear.

Ex:- Determine if the system described by the following i/p-o/p eq^n is linear or non-linear.

i> $y(n) = x(n) + \frac{1}{x(n-1)}$ ii> $y(n) = x^2(n)$

iii> $y(n) = n\,x(n)$

i) $y(n) = x(n) + \frac{1}{x(n-1)}$

$y_1(n) = T[x_1(n)] = x_1(n) + \frac{1}{x_1(n-1)}$.

$y_2(n) = T[x_2(n)] = x_2(n) + \frac{1}{x_2(n-1)}$.

$y_3(n) = T[a_1 y_1(n) + a_2 y_2(n)]$

$= T[a_1 x_1(n) + a_2 x_2(n) + \frac{1}{a_1 x(n-1) + a_2 x_2(n-1)}]$ —— (1)

$a_1 y_1(n) + a_2 y_2(n) = a_1 x_1(n) + \frac{a_1}{x(n-1)} + a_2 x_2(n) + \frac{a_2}{x_2(n-1)}$ (2)

eqn (1) & (2) are not equal.

As the eqn are not equal, so the system is non-Linear.

ii) $y(n) = x^2(n)$

$y_1(n) = T[x_1(n)] = x_1^2(n)$

$y_2(n) = T[x_2(n)] = x_2^2(n)$

The weighted sum of o/ps is

$a_1 T[x_1(n)] + a_2 T[x_2(n)]$

$= a_1 x_1^2(n) + a_2 x_2^2(n)$ —— (1)

The o/p

$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)]$

$= [a_1 x_1(n) + a_2 x_2(n)]^2$ —— (2)

Eqn (1) $\neq$ eqn (2)

$\rightarrow$ non linear.

ii) $y(n) = n\,x(n)$

$$y_1(n) = T[x_1(n)] = n\,x_1(n)$$
$$y_2(n) = T[x_2(n)] = n\,x_2(n)$$

R.H.S $= a_1 T[x_1(n)] + a_2 T[x_2(n)] = a_1 n x_1(n) + a_2 n x_2(n)$ .... (1)

L.H.S $= T[a_1 x_1(n) + a_2 x_2(n)] = n a_1 x_1(n) + n a_2 x_2(n)$ --- (ii)

∴ eqn (1) = eqn (ii)

∴ the system is linear

iv) Time Varriant & Time-Invarient system

Time Invarient

A system is said to be time-invarient or shift invarient if the characteristice of the system don't change with time.

Ex → $y(n) = x(n)$

$y(n,k) = T[x(n-k)]$

i.e the i/p sequence is shifted by k samples

$y(n,k) = y(n-k)\ \forall\ k$

→ If the o/p signal of a system shifts k units of time upon delaying the i/p signal by k units, then the system is time invariant system.

Ex → $y(n) = x(n) + x(n-1)$

Time Varriant

The system is time-varriant if ~~$y(n,k) \neq x(n-k)$~~ the o/p is $y(n,k) \neq y(n-k)$

Q// Determine the following systems are time variant or time-invariant:

Ans i) $y(n) = x(n) + x(n-1)$ ii) $y(n) = x(-n)$

Ans

$$y(n) = T[x(n)] = x(n) + x(n-1)$$

$$y(n,k) = T[x(n-k)] = x(n-k) + x(n-k-1)$$

$$y(n-k) = x(n-k) + x(n-k-1)$$

Hence $y(n,k) = y(n-k)$

So, it is time-invariant.

ii) $y(n) = x(-n)$

$$y(n) = T[x(n)] = x(-n).$$

$$y(n,k) = T[x(n-k)] = \cancel{x(-(n-k))}\ x(-n-k)$$

$$y(n-k) = x(-(n-k)) = x(-n+k)$$

$$y(n,k) \neq y(n-k)$$

So, the system is time-variant.

Q// i) $y(n) = x\left(\frac{n}{2}\right)$

Ans

$$y(n) = T[x(n)] = x\left(\frac{n}{2}\right).$$

$$y(n,k) = T[x(n-k)] = x\left(\frac{n-k}{2}\right)$$

$$y(n-k) = x\left(\frac{n-k}{2}\right)$$

$y(n) = \cos x(n)$

ii) $y(n) = n x^2(n)$.

$$y(n,k) = T[x(n-k)] = (n-k)x^2(n-k)$$
$$y(n-k) = (n-k)x^2(n-k)$$

$$\xrightarrow{x(n)} \boxed{T} \xrightarrow{y(n) = T[x(n)]}$$

Discrete-time system representation.

Impulse Response

If the i/p to the system is a unit impulse i.e, $x(n) = \delta(n)$ then the o/p of the system is Known as impulse response denoted by $h(n)$.

$$\boxed{h(n) = T[\delta(n)]}$$

3.5 **FIR & IIR System**

~~FIR Systems~~

LTI System

An LTI system is causal if & only if its impulse response is zero for negative values of n.

FIR

If the impulse response of the system is of finite duration, then the system is called a ~~FIR~~ finite impulse response (FIR).

Ex :- $h(n) = \begin{cases} 1, & \text{for } n = -1, 2 \\ 2, & \text{"} \ n = 1 \\ 3, & \text{"} \ n = 0, 3 \\ 0 & \text{otherwise} \end{cases}$

~~IIR~~ IIR.

An infinite impulse response system has an impulse response for finite duration.

Ex :- $h(n) = a^n u(n)$ —— (I)

Q.34 - Explain the properties of convolution & interconnection of LTI system.

→ Properties of convolution

i) Commutative Law : $x(n) * h(n) = h(n) * x(n)$.

ii) Associative Law : $[x(n) * h_1(n)] * h_2(n)$

$= x(n) * [h_1(n) * h_2(n)]$

iii) Distributive Law : $x(n) * [h_1(n) + h_2(n)]$

$= x(n) * h_1(n) + x(n) * h_2(n)$

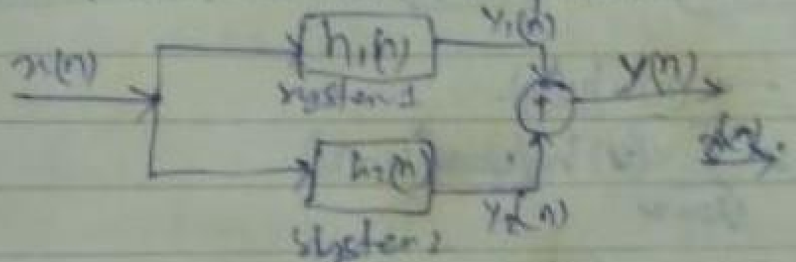
Interconnection of LTI systems:-

i) Parallel connection of systems

ii) Cascade " of two " "

i) Parallel connection of systems.

Consider two LTI systems with impulse response $h_1(n)$ & $h_2(n)$ connected in parallel.



o/p of system 1 is

$y_1(n) = x(n) * h_1(n)$.

o/p of system 2 is

$y_2(n) = x(n) * h_2(n)$.

$y(n) = y_1(n) + y_2(n)$

$= x(n) * h_1(n) + x(n) * h_2(n)$.

$= \sum_{k=-\infty}^{\infty} x(k) h_1(n-k) + \sum_{k=-\infty}^{\infty} x(k) h_2(n-k)$

$$= \sum_{K=-\infty}^{\infty} x(K)\left[h_1(n-K) + h_2(n-K)\right]$$

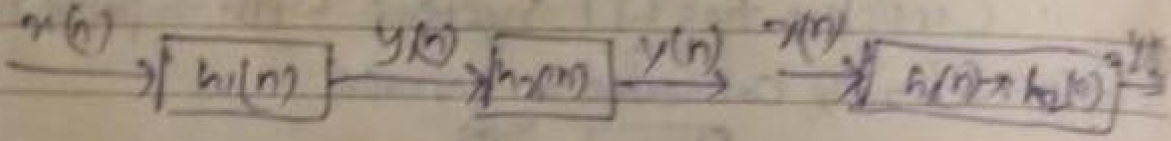
$$= \sum_{K=-\infty}^{\infty} x(K)\, h(n-K)$$

$$= x(n) * h(n)$$

where $h(n) = h_1(n) + h_2(n)$.

ii) Cascade Connection of two system

Consider two LTI systems with impulse response $h_1(n)$ & $h_2(n)$ connected in cascade.



$$y_1(K) = x(K) * h_1(K).$$

$$= \sum_{v=-\infty}^{\infty} x(v)\, h_1(K-v)$$

$$y(n) = y_1(K) * h_2(K).$$

$$= \left[\sum_{v=-\infty}^{\infty} x(v)\, h_1(K-v)\right] * h_2(K).$$

$$= \sum_{K=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} x(v)\, h_1(K-v)\, h_2(n-K)$$

Let $K - v = p$

$$y(n) = \sum_{v=-\infty}^{\infty} x(v) \sum_{p=-\infty}^{\infty} h_1(p)\, h_2(n-v-p)$$

$$= \sum_{v=-\infty}^{\infty} x(v)\, h(n-v)$$

$$= x(n) * h(n).$$

Chapter-3

3. The Z-Transform & Its application to the Analysis of LTI Systems :->

3.1 Discuss Z-transform & its application to LTI System :->

Introduction

i) Z-transform technique is an important tool in the analysis of Signal & LTI Systems.

ii) The Z-transform plays the same role in the analysis of discrete time Signals & LTI Systems as the Laplace transform does in the analysis of Continuous time Signals & LTI Systems.

iii) The Z-transform is used for the frequency analysis of a Linear time Variant discrete domain Calculation.

Defination of Z-transform

3.1.1 State & explain direct Z-transform :->

The Z-transform of a discrete time Signal $x(n)$ is defined by the power Series as

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}} \quad \text{——— (1)}$$

Where Z is a Complex Variable

Eqn (1) is called as direct Z-transform & also called as two sided Z-transform.

→ The Z-transform of the Signal is denoted by

$$X(z) = Z[x(n)]$$

or $x(n) \xleftrightarrow{Z} X(z)$

Region of Convergence (ROC)

The set of all values of z in the z-plane for which the magnitude of X(z) is finite, is called the Region of Convergence (ROC).

characteristics of ROC

i) The ROC is a ring or disc in the z-plane centred at the origin.

ii) The ROC doesn't contain any poles.

iii) If x(n) is a causal sequence i.e., $x(n)=0,\ n<0$, the ROC is entire z-plane except $z=0$

iv) If x(n) is a non-causal sequence i.e., $x(n)=0$, for $n\geq 0$, the ROC is entire the z-plane except $z=\infty$.

Causal Signal

If x(n) is a causal signal i.e, $x(n)=0, n<0$, then

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

→ The z-transform of the causal sequence consists of negative powers of z.

Non-Causal Signal

If x(n) is a non-causal sequence i.e, $x(n)=0$ for $n\geq 0$ then the z-transform is $X(z) = \sum_{n=-\infty}^{-1} x(n) z^{-n}$

→ The z-transform of a non-causal consists of positive powers of z.

Example 1

Q) Find the z transform of the following finite duration signal.

$$x(n) = \{\underset{\uparrow}{1}, 0, 3, -1, 2\}$$

Soln

$$x(n) = \{\underset{\uparrow}{1}, 0, 3, -1, 2\}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \cancel{x(0) z^{-1}}$$

$$X(z) = \sum_{n=0}^{4} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4}$$

$$= 1 \times 1 + 0 \times z^{-1} + 3 \times z^{-2} + (-1) \times z^{-3} + 2 \times z^{-4}$$

$$= 1 + 3z^{-2} - z^{-3} + 2z^{-4}$$

Example 2 Determine the z-transform of the following finite duration sequence

a) $x(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$

b) $x(n) = \{1, 2, 5, 7, 0, \underset{\uparrow}{1}\}$

c) $x(n) = \{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\}$

d) $x(n) = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\}$

e) $x(n) = \delta(n)$

(f) $x(n) = \delta(n-k)$, $k > 0$

(g) $x(n) = \delta(n+k)$, $k < 0$

Sol (a)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$$

$$X(z) = \sum_{n=0}^{5} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + x(5) z^{-5}$$

$$= 1\times1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + 0\cdot z^{-4} + 1\cdot z^{-5}$$

$$= 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

(b) $x(n) = 1+2$

ROC : Entire z-plane except $z=0$

(b) $x(n) = \{1, 2, 5, 7, 0, \underset{\uparrow}{1}\}$

$$X(z) = \sum_{n=-5}^{-1} x(n) z^{-n}$$

$$= x(-5) z^{5} + x(-4) z^{4} + x(-3) z^{3} + x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0}$$

$$= 1\cdot z^{5} + 2z^{4} + 5\cdot z^{3} + 7z^{2} + 0\cdot z^{1} + 1\cdot 1$$

$$= z^{5} + 2z^{4} + 5z^{3} + 7z^{2} + 1$$

ROC : Entire z-plane except $z=\infty$

(c) $x(n) = \{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\}$

$$X(z) = \sum_{n=0}^{7} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + x(5) z^{-5} + x(6) z^{-6} + x(7) z^{-7}$$

$$= 0\times1 + 0\times z^{-1} + 1\times z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + 0\times z^{-6} + z^{-7}$$

$$= z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$$

ROC : Entire z-plane except $z=0$

(d) $x(n) = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\}$

$$X(z) = \sum_{-2}$$ $(\because z^0 = 1)$

$$X(z) = \sum_{n=-2}^{3} x(n) z^{-n}$$

$$= x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$= 2z^2 + 4z + 5 + 7z^{-1} + z^{-3}$$

(e) $x(n) = \delta(n)$

Given $\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \cdots + x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

$$= \cdots + 0z^2 + 0z^1 + 1 \cdot z^0 + 0 \cdot z^{-1} + 0 \cdot z^{-2} + \cdots$$

$$= 1$$

or

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

$$= \delta(0) z^0 = 1$$

(f) $x(n) = \delta(n-k)$, $k > 0$

$$\delta(n) = \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases}$$

~~let n-k = m -> n~~

$$\delta(n-k) = \begin{cases} 1, & \text{for } n = k \\ 0, & \text{for } n \neq k \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n-k) z^{-k}$$

$$= 1 \times z^{-k} = z^{-k}$$

ROC: Entire z-plane except z=0

(9) $x(n) = \delta(n+k)$

Let $m = n+k$

$n = m-k$

$$X(z) = \sum_{m=-\infty}^{\infty} \delta(m) z^{-(m-k)}$$

$$= \sum_{m=-\infty}^{\infty} \delta(m) z^{-m} \cdot z^{k}$$

$$= z^{k} \sum_{m=-\infty}^{\infty} \delta(m) z^{-m}$$

$$= z^{k} \cdot 1 = z^{k}$$

ROC : Entire z-plane except z=0

Q11 Determine the z-transform & ROC of the signal $x(n) = a^n u(n)$.

Soln The given signal is causal & of infinite duration.

The z-transform of $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= (az^{-1})^0 + (az^{-1})^1 + (az^{-1})^2 + (az^{-1})^3 + \cdots$$

$$= 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$$

$$= \frac{1}{1-az^{-1}} \quad \left[1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a} \right]$$

$|az^{-1}| < 1 \Rightarrow |z| > a$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \quad \text{ROC is } |z|>a$$

The ROC is the exteriore of a Circle having radius $|a|$ as shown in fig ①

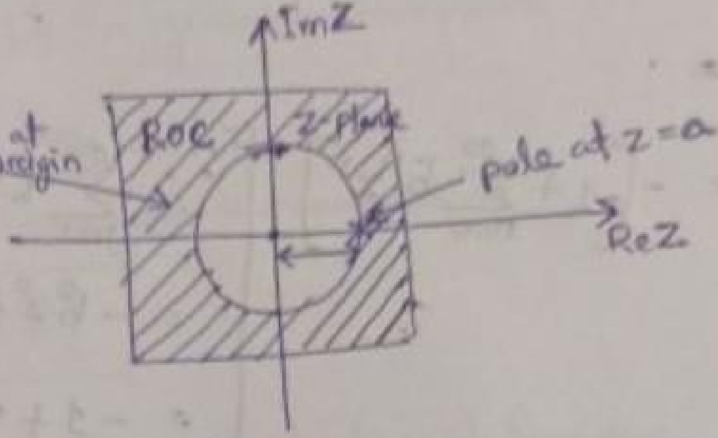


Fig ① ROC

Q/ Find the z-transform & the ROC of the signal $x(n) = -b^n u(-n-1)$.

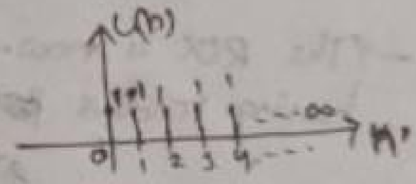
Soln

$(n+2) \to$ right shift
$(n-2) \to$ left shift

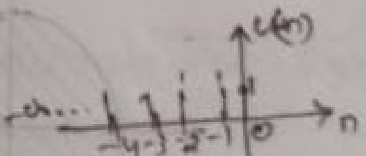
The given signal is of infinite duration and anti causal.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

We know that $u(n) = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$



$$u(-n) = \begin{cases} 1, & n \leq 0 \\ 0, & n > 0 \end{cases}$$



So $u(-(n+1)) = u(-n-1) = \begin{cases} 0, & n \geq 0 \\ 1, & n \leq -1 \end{cases}$

$$= \begin{cases} 0, & n \geq 0 \\ b^n, & n \leq -1 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [-b^n \cdot u(-n-1)] z^{-n}$$

$$= \sum_{n=-\infty}^{-1} [-b^n u(-n-1)] z^{-n} + \sum_{n=0}^{\infty} [-b^n u(-n-1)] z^{-n} \to 0$$

$$= \sum_{n=-\infty}^{-1} -b^n \cdot z^{-n} = -\sum_{n=-\infty}^{-1} b^n z^{-n}$$

multiplying (-)

$$= -\sum_{n=1}^{\infty} b^{-n} z^{n}$$

$$\left[\because \sum_{n=-\infty}^{-1} b^{n} z^{-n} = \dots + b^{-4}z^{4} + b^{-3}z^{3} + b^{-2}z^{2} + b^{-1}z^{1} = \sum_{n=1}^{\infty} b^{-n} z^{n}\right]$$

$$= -\left[-1 + \sum_{n=0}^{\infty} b^{-n} z^{n}\right] \quad \left[\because -\sum_{n=1}^{\infty} b^{-n} z^{n} = -\left[-b^{0}z^{0} + \sum_{n=0}^{\infty} b^{-n} z^{n}\right] = -\left[-1 + \sum_{n=0}^{\infty} b^{-n} z^{n}\right]\right]$$

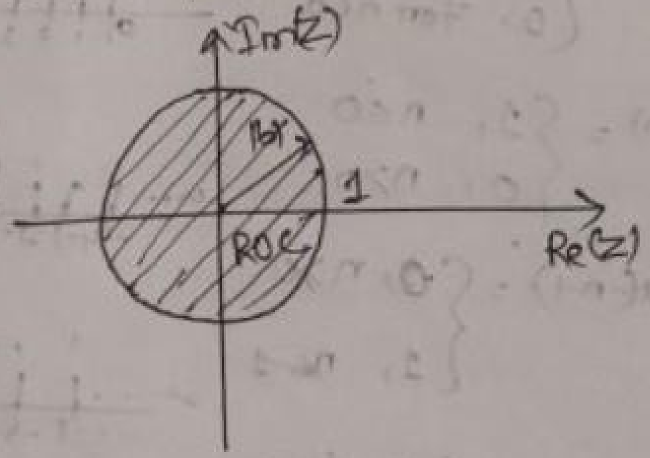
$$= 1 - \sum_{n=0}^{\infty} (b^{-1}z)^{n}$$

$$\frac{1}{1-\frac{z}{b}}$$

$$= 1 - \frac{1}{1-b^{-1}z} = 1 - \frac{1}{1-\frac{z}{b}}$$

Hence ROC is $\left|\frac{z}{b}\right| < 1 \Rightarrow |z| < |b|$

→ The ROC is now the interior of a circle having radius $|b|$.



5.12 State & explain inverse Z-transform :→

The inverse z-transform is a process of determining the sequence which generates a given z-transform.

It is ~~denoted~~ expressed by

The inverse z-transform is expressed as

$$\boxed{x(n) = Z^{-1}[X(z)]}$$

There are basically three methods used for the evaluation of inverse z-transform.

1. Long division method or Power Series method
2. Partial fraction Expansion method
3. Residue method

Long division method :→

The z-transform is expressed as

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

We know that the z-transform of a discrete-time-signal may be expressed as the ratio of two polynomials in z.

$$X(z) = \frac{N(z)}{D(z)}$$

The above ratio of polynomial for z-transform may be divided out to produce a power series in form of an eqn with the co-efficients representing the sequence values in the time domain as

$$X(z) = \frac{N(z)}{D(z)} = \sum_{n=0}^{\infty} a_n z^{-n} = a_0 z^0 + a_1 z^{-1} + a_2 z^{-2} + \cdots$$

Here the co-efficients a_0, a_1, a_2 are the values of $x(n)$

Partial fraction Expansion method

Q) Find the inverse z-transform of

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

Soln

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

$$= \frac{1}{1-\frac{3}{2z}+\frac{1}{2z^2}}$$

$$= \frac{1}{\frac{2z^2-3z+1}{2z^2}}$$

$$= \frac{2z^2}{2z^2-3z+1}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{2z}{2z^2-3z+1}$$

$$= \frac{2z}{(z-1)(2z-1)}$$

$$= \frac{A}{z-1} + \frac{B}{2z-1}$$

$$= \frac{A(2z-1)+B(z-1)}{(z-1)(2z-1)}$$

$$\Rightarrow 2z = A(2z-1)+B(z-1)$$

$$= A2z - A + Bz - B$$

$$= z(2A+B) - (A+B)$$

$$z(2A+B) = 2z$$

$$\Rightarrow 2A+B = 2$$

$$\begin{array}{r} A+B = 0 \\ (-)\;\;(-) \\ \hline A = 2 \end{array}$$

$2A+B=2$

$\Rightarrow 2\times 2+B=2$

$\Rightarrow 4+B=2$

$\Rightarrow B=-2$

$$\frac{X(z)}{z}=\frac{2}{(z-1)}-\frac{2}{(2z-1)}$$

$$X(z)=2\left[\frac{z}{(z-1)}\right]-2\left[\frac{z}{(2z-1)}\right]$$

$$=2\left(\frac{z}{z-1}\right)-\left(\frac{z}{z-1/2}\right)$$

$$Z[u(n)]=\frac{z}{z-1}$$

Taking inverse z-transform both the sides

$$z^{-1}[X(z)]=2z^{-1}\left[\frac{z}{z-1}\right]-z^{-1}\left(\frac{z}{z-1/2}\right)$$

$$\Rightarrow x(n)=2u(n)-\left(\tfrac{1}{2}\right)^n u(n) \text{ Ans}$$

$\Rightarrow \cancel{x(n) = 2} -$

Q11 Find the inverse z-transform of the following z-transform

$$X(z)=\frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$$

Soln

$$X(z)=\frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$$

$$=\frac{z(z+3)}{z^2+3z+2}$$

$$\Rightarrow \frac{X(z)}{z}=\frac{z+3}{z^2+3z+2}$$

$$=\frac{z+3}{(z+1)(z+2)}$$

Chapter-5

Introduction

i) The fast fourier transform (FFT) doesn't represent a transform different from DFT but they are special algorithm for speedier implementation of DFT.

ii) The FFT is computationally efficient algorithms used for evaluating the DFT.

iii) FFT requires comparatively smaller no. of arithmetic operations such as multiplications & additions than DFT.

iv) FFT also requires lesser computational time than DFT.

v) The FFT is based on decomposition & breaking the transform into smaller transforms & combining them to get the total transform.

vi) This algorithm is used for computing DFT when the size N is a power of 2 and power of 4.

vii) FFT computation technique is used in digital spectral analysis, filter simulation, auto correlation & pattern recognition.

viii) FFT reduces the computation techniques used to compute a discrete fourier transform & improves the performance by a factor 100 or more over direct ~~evaluated~~ DFT evaluation.

Direct computation of DFT :→

Let $x(n)$ is a complex valued sequence. Then N-point DFT is

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}, \quad 0 \le k \le N-1 \qquad \left[W_N^{nk} = e^{\frac{-j2\pi kn}{N}} = \cos\frac{2\pi kn}{N} - j\sin\frac{2\pi kn}{N} \right]$$

$$\Rightarrow X_R(K) + jX_I(K) = \sum_{n=0}^{N-1} \{x_R(n) + jx_I(n)\}\left\{\cos\frac{2\pi kn}{N} - j\sin\frac{2\pi kn}{N}\right\} \quad \text{—(1)}$$

From eqn (1)

$$X_R(K) = \sum_{n=0}^{N-1} \left[x_R(n)\cos\frac{2\pi kn}{N} + x_I(n)\sin\frac{2\pi kn}{N}\right] \quad \text{—(2)}$$

$$X_I(K) = \sum_{n=0}^{N-1} \left[x_R(n)\cos\frac{2\pi kn}{N} + x_I(n)\sin\frac{2\pi kn}{N}\right] \quad \text{—(3)}$$

The direct computation of the eqn (3) requires

1. 4N real multiplications for each value of k.
2. (4N-2) real additions for N values of k.
3. $4N^2$ real multiplications for N values of k.
4. N(4N-2) read additions for N values of k.

Properties of FFT

Symmetry property: $W_N^{k+N/2} = -W_N^k$

Periodicity property: $W_N^{k+N} = W_N^k$

Radix of FFT Algorithm :→

In an N-point sequence, if N can be expressed as $N = r^m$, where m is an integer, then the sequence can be decimated into r-point sequences.

→ In computing N-point DFT by this method, the no. of stages of computation will be m times.

→ The number "r" is called the Radix of FFT algorithm.

Radix-2 Algorithm (r=2)

In radix-2 FFT algorithm, the o/p points N can be expressed as a power of 2 i.e, $N = 2^m$, where m is an integer.

→ Hence the decimation (decomposition) can be performed m times where $m = \log_2 N$.

→ Total no. of addition → $N \log_2 N$

→ Total no. of multiplication → $\frac{N}{2} \log_2 N$

	DFT	FFT
① Addtn	→ N(N-1)	→ $N \log_2 N$
② multn	→ N^2	→ $N/2 \log_2 N$

Scanned with CamScanner

There are basically two classes of FFT Computation :→

① Decimation-in-Time FFT algorithms

② Decimation-in-Frequency FFT algorithms

Decimation means decomposition into decimal parts.

DIT Algorithms :-

Let $x(n)$ is an N-point sequence, where N is assumed to be power of 2. i.e, radix-2.

→ Decimate this sequence into two sequences of length N/2, where one sequence consists of the even-indexed values of $x(n)$ & the other of odd-indexed values of $x(n)$

i.e, $x_e(n) = x(2n) \qquad n = 0, 1, 2, \cdots \frac{N}{2} - 1$

$x_o(n) = x(2n+1) \qquad n = 0, 1, \cdots \frac{N}{2} - 1$

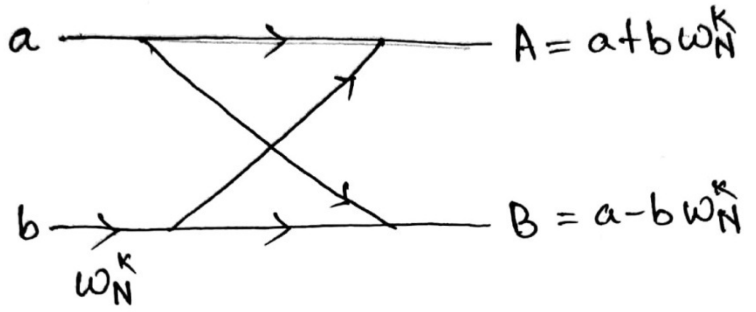
Steps Radix-2 DIT FFT Alogrithm

1. The no. of i/p samples $N = 2^M$, where M is an integer.
2. The i/p sequence is shuffled through "bit-REVERSAL".
3. The no. of stages in the flowgraph is given by $M = \log_2 N$
4. Each stage has $\frac{N}{2}$ butterflies
5. Input/outputs for each butterfly are separated by 2^{m-1} samples, where m represents the stage index, i.e, for stage-1 $m = 1$ stage-2, $m = 2$, so on.
6. the no. of complex multiplication is given by $\frac{N}{2}\log_2 N$
7. The no. of complex addition is given by $N\log_2 N$.
8. The twiddle factor exponents are a function of the stage index 'm' & is given by, $\boxed{K = \frac{Nt}{2^m}}$, $t = 0, 1, 2, \cdots, 2^{m-1} - 1$
9. the no. of sets or sections of butterflies in each stage is given by the formula 2^{M-m}.

~~10. the~~

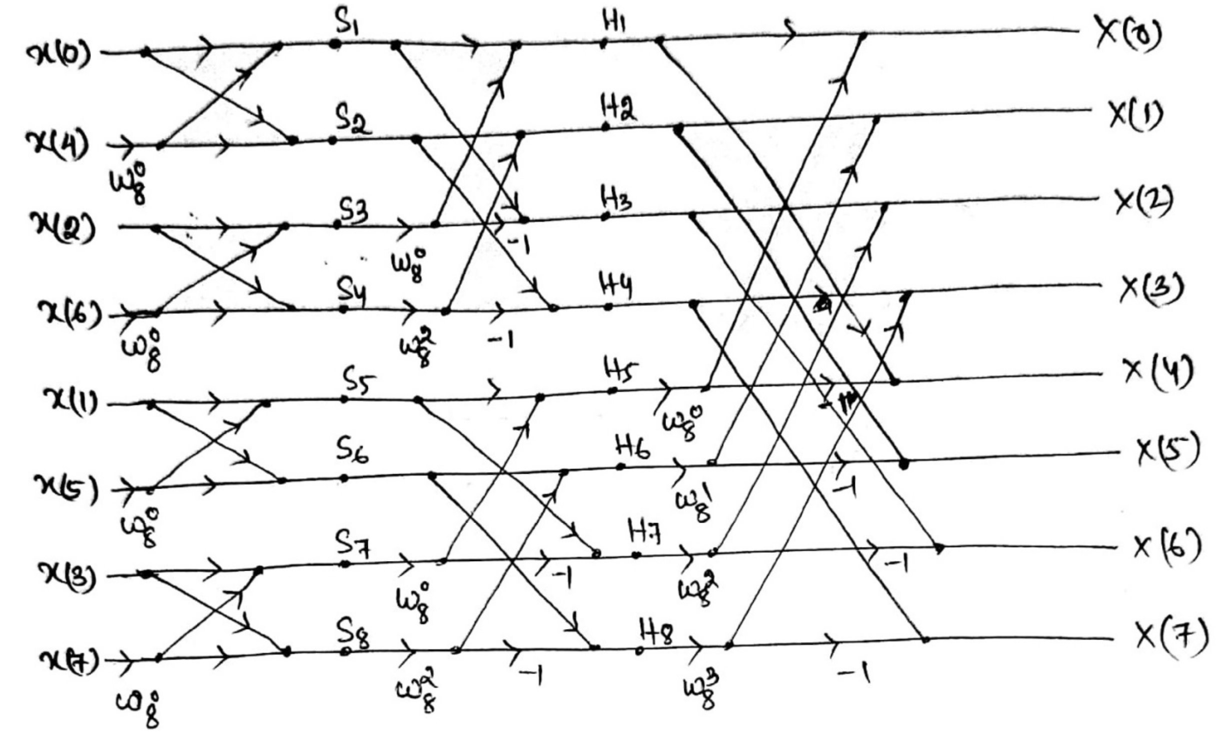
10. The exponent repeat factor (ERF), which is the no. of times the exponent sequence associated with 'm' is repeated is given by $2^{(M-m)}$.

Butterfly diagram for DIT Algorithm



Ex

Computation Procedure to find DFT of 8-point Sequence :- (DIT algorithm)



1. Draw the ~~Flow~~ Butterfly diagram for N = 8 shown in fig. 6.6.
2. Find the values of twiddle factors $W_8^0, W_8^1, W_8^2, W_8^3$.
3. Compute the values at the o/p of stage 1 as shown in table below. These o/p become the i/ps to stage 2.
4. Compute the values at the o/p of stage 2 as shown in table below. These o/p become inputs to stage 3.
5. Compute the values at the o/p of stage 3 as shown below. These values represents the DFT values of $x(n)$.

Input	o/p of stage-1	o/p of stage-2	output
$x(0)$	$S_1 = x(0) + W_8^0 x(4)$	$H_1 = S_1 + W_8^0 S_3$	$X(0) = H_1 + W_8^0 H_5$
$x(4)$	$S_2 = x(0) - W_8^0 x(4)$	$H_2 = S_2 + W_8^2 S_4$	$X(1) = H_2 + W_8^1 H_6$
$x(2)$	$S_3 = x(2) + W_8^0 x(6)$	$H_3 = S_1 - W_8^0 S_3$	$X(2) = H_3 + W_8^2 H_7$
$x(6)$	$S_4 = x(2) - W_8^0 x(6)$	$H_4 = S_2 - W_8^2 S_4$	$X(3) = H_4 + W_8^3 H_8$
$x(1)$	$S_5 = x(1) + W_8^0 x(5)$	$H_5 = S_5 + W_8^0 S_7$	$X(4) = H_1 - W_8^0 H_5$
$x(3)$	$S_6 = x(1) - W_8^0 x(6)$	$H_6 = S_6 + W_8^2 S_8$	$X(5) = H_2 - W_8^1 H_6$
$x(5)$	$S_7 = x(3) + W_8^0 x(7)$	$H_7 = S_5 - W_8^0 S_7$	$X(6) = H_3 - W_8^2 H_7$
$x(7)$	$S_8 = x(3) - W_8^0 x(7)$	$H_8 = S_6 - W_8^2 S_8$	$X(7) = H_4 - W_8^3 H_8$

Q11 Compute the DFT of the sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ using DIT FFT algorithm.

Soln We know that $W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$

Here $N = 8$

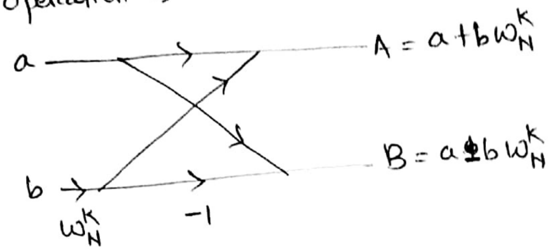
$$W_8^0 = e^{-j\left(\frac{2\pi}{8}\right)0} = 1$$

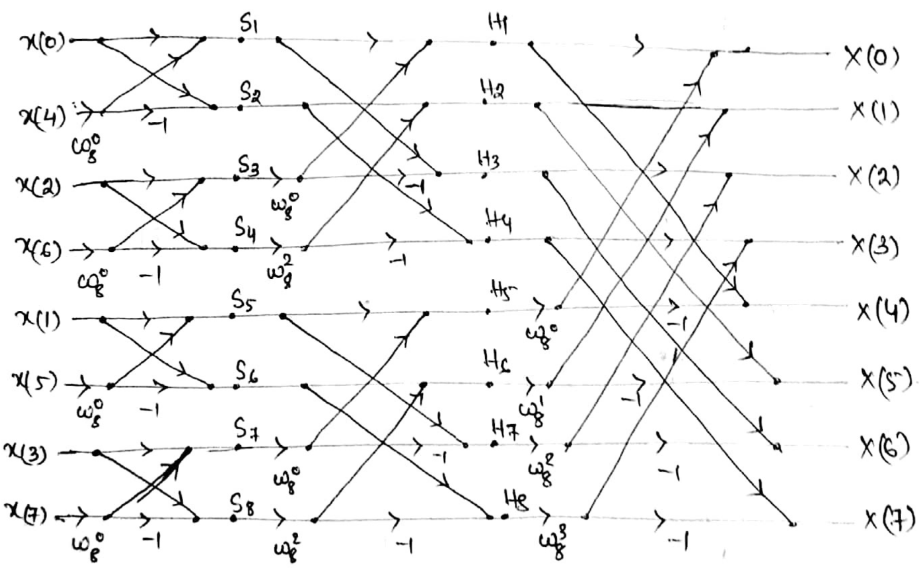
$$W_8^1 = e^{-j\frac{2\pi}{8}} = e^{-j\pi/4} = \cos\pi/4 - j\sin\pi/4 = 0.707 - j0.707$$

$$W_8^2 = e^{-j\left(\frac{2\pi}{8}\right)\times 2} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = -j$$

$$W_8^3 = e^{-j\left(\frac{2\pi}{8}\right)3} = e^{-j3\pi/4} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - j0.707$$

The basic operation is



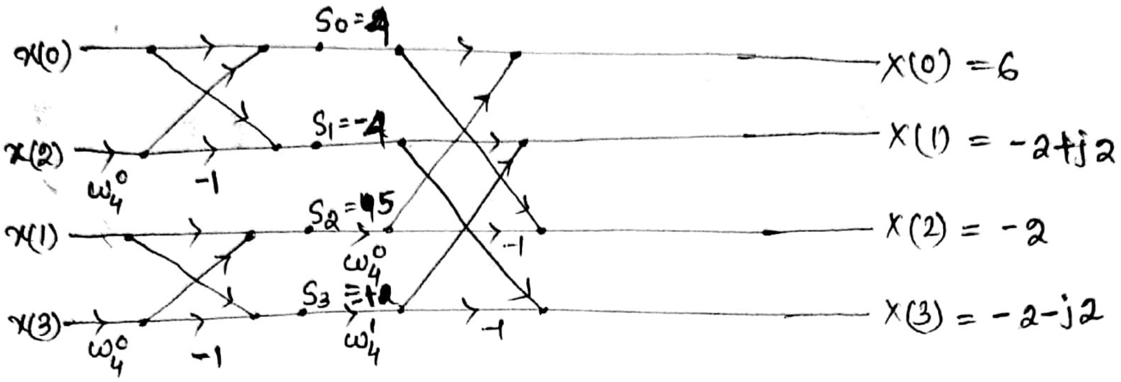


Input	o/p of stage-1	o/p of stage-2	o/p
$x(0)$	$S_1 = x(0) + W_8^0 x(4) = 1+4 = 5$	$H_1 = S_1 + W_8^0 S_3 = 5+5 = 10$	$X(0) = H_1 + W_8^0 H_5 = 10+10 = 20$
$x(4)$	$S_2 = x(0) - W_8^0 x(4) = 1-4 = -3$	$H_2 = S_2 + W_8^2 S_4 = -3+j(1) = -3-j$	$X(1) = H_2 + W_8^1 H_6 = (-3-j) + (0.707-j0.707)(-1-3j) = -5.828 - j2.414$
$x(2)$	$S_3 = x(2) + W_8^0 x(6) = 3+2 = 5$	$H_3 = S_1 - W_8 S_3 = 5-5 = 0$	$X(2) = H_3 + W_8^2 H_7 = 0 + (-j)0 = 0$
$x(6)$	$S_4 = x(2) - W_8^0 x(6) = 3-2 = 1$	$H_4 = S_2 - W_8^2 S_4 = -3-(-j)1 = -3+j$	$X(3) = H_4 + W_8^3 H_8 = (-3+j) + (-0.707-j0.707)(-1+3j) = -0.172 - j0.414$
$x(1)$	$S_5 = x(1) + W_8^0 x(5) = 2+3 = 5$	$H_5 = S_5 + W_8^0 S_7 = 5+5 = 10$	$X(4) = H_1 - W_8^0 H_5 = 10-10 = 0$
$x(5)$	$S_6 = x(1) - W_8^0 x(5) = 2-3 = -1$	$H_6 = S_6 + W_8^2 S_8 = -1+3(-j) = -1-3j$	$X(5) = H_2 - W_8^1 H_6 = (-3-j) - (0.707-j0.707)(-1-2j) = -0.172 + j0.414$
$x(3)$	$S_7 = x(3) + W_8^0 x(7) = 4+1 = 5$	$H_7 = S_5 - W_8^0 S_7 = 5-5 = 0$	$X(6) = H_3 - W_8^3 H_8 = (-3+j) - (-0.707-j0.707)(-1+3j) = -5.828 + j2.414$
$x(7)$	$S_8 = x(3) - W_8^0 x(7) = 4-1 = 3$	$H_8 = S_6 - W_8^2 S_8 = -1-3(-j) = -1+3j$	

$$X(k) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, -5.828 + j2.414\}$$

Q11 Compute the 4-point DIT-FFT of the following sequence

$$x(n) = \{0, 2, 4, 3\}$$



Input	O/P stage-1	O/P
$x(0) = 0$	$S_0 = x(0) + W_4^0 x(2)$ $= 0 + 1 \times 4 = 4$	$X(0) = S_0 + W_4^0 S_2$ $= 4 + 1 \times 5 = 9$
$x(2) = 2$	$S_1 = x(0) - W_4^0 x(2)$ $= 0 - 1 \times 4 = -4$	$X(1) = S_1 + W_4^1 S_3$ $= -4 + (-j)(-1)$ $= -4 + j$
$x(1) = 1$	$S_2 = x(1) + W_4^0 x(3)$ $= 2 + 1 \times 3 = 5$	$X(2) = S_0 - W_4^0 S_2$ $= 4 - 1 \times 5 = -1$
$x(3) = 3$	$S_3 = x(1) - W_4^0 x(3)$ $= 2 - 1 \times 3 = -1$	$X(3) = S_1 - W_4^1 S_3$ $= -4 - (-j)(-1)$ $= -4 - j$

$$N = 4 = 2^2$$

We know $W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$ $\quad k = 0, 1$

$$W_4^0 = e^{-j\left(\frac{2\pi}{4}\right)0} = e^0 = 1$$

$$W_4^1 = e^{-j\left(\frac{2\pi}{4}\right)1} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = -j$$

$X(K) = \{9, -4+j, -1, -4-j\}$ Ans

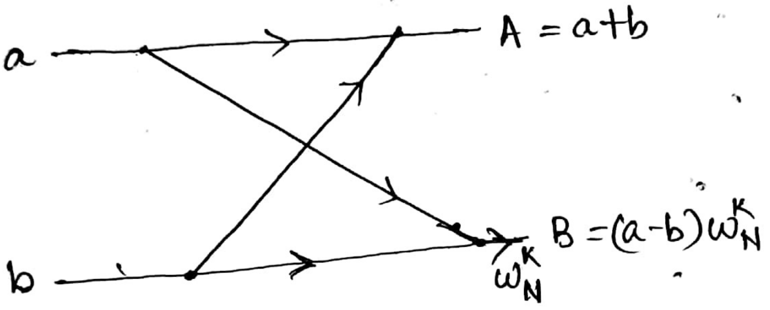
Decimation in frequency algorithm → (DIF)

→ In DIF algorithm, the i/p sequence $X(K)$ is divided into smaller & smaller subsequence.

→ In this algorithm the i/p sequence $x(n)$ is pattern into 2 sequences, each of length N/2 samples.

→ The 1st sequence $x_1(n)$ consists of 1st N/2 samples of $x(n)$ & the 2nd sequence $x_2(n)$ consists of the last N/2 samples.

Butterfly diagram of DIF

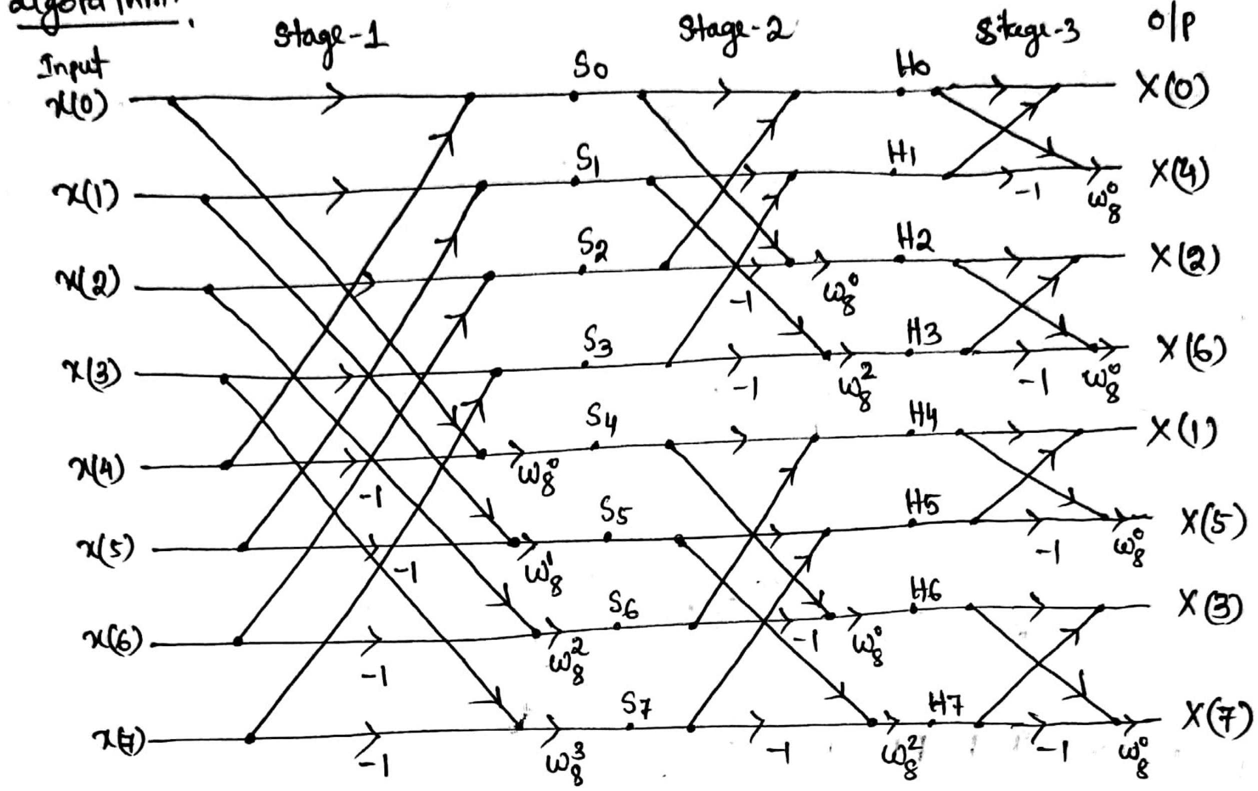


Steps Radix-2 DIF-FFT Algorithms

1. The no. of i/p samples $N = 2^M$, where M = the no. of stages.
2. The i/p sequence is in natural order. (No Bit-Reversal is required)
3. The no. of stages in the flow graph is given by $M = \log_2 N$
4. Each stage consists of $\frac{N}{2}$ butterflies.
5. Inputs/outputs for each butterfly are separated by 2^{M-m} samples, where m represents the stage index i.e, for first stage, m = 1, for second stage m = 2, and so on...
6. The ~~total~~ no. of complex multiplications is given by $\frac{N}{2}\log_2 N$.
7. The no. of complex additions is given by $N\log_2 N$.
8. The no. of sets of butterflies in each stage is given by the formula 2^{m-1}
9. The twiddle factor exponents are a function of the stage index (m) is given by $k = \frac{Nt}{2^{M-m+1}}$ where $t = 0, 1, 2, \cdots 2^{M-m}$

10. The exponent repeat factor (ERF), which is the no. of times the exponent sequences associated with m is repeated is given by 2.

Computation Procedure to find DFT of 8-point sequence using DIF algorithm.



Input	o/p of stage-1	o/p of stage-2	o/p
$x(0)$	$S_0 = x(0)+x(4)$	$H_0 = S_0+S_2$	$X(0) = H_0+H_1$
$x(1)$	$S_1 = x(1)+x(5)$	$H_1 = S_1+S_3$	$X(4) = [H_0-H_1]W_8^0$
$x(2)$	$S_2 = x(2)+x(6)$	$H_2 = [S_0-S_2]W_8^0$	$X(2) = H_2+H_3$
$x(3)$	$S_3 = x(3)+x(7)$	$H_3 = [S_1-S_3]W_8^2$	$X(6) = [H_2-H_3]W_8^0$
$x(4)$	$S_4 = [x(0)-x(4)]W_8^0$	$H_4 = S_4+S_6$	$X(1) = H_4+H_5$
$x(5)$	$S_5 = [x(1)-x(5)]W_8^1$	$H_5 = S_5+S_7$	$X(5) = [H_4-H_5]W_8^0$
$x(6)$	$S_6 = [x(2)-x(6)]W_8^2$	$H_6 = [S_4-S_6]W_8^0$	$X(3) = H_6+H_7$
$x(7)$	$S_7 = [x(3)-x(7)]W_8^3$	$H_7 = [S_5-S_7]W_8^2$	$X(7) = [H_6-H_7]W_8^0$

$$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$

Problems

Q1// Find the DFT of the sequence $x(n) = \{1,2,3,4,4,3,2,1\}$ using DIF FFT algorithm.

Soln $x(n) = \{1,2,3,4,4,3,2,1\}$

Here $N = 8 = 2^3$

We know $W_N^K = e^{-j\left(\frac{2\pi}{N}\right)K}$

$K = 0, 1, 2, 3$

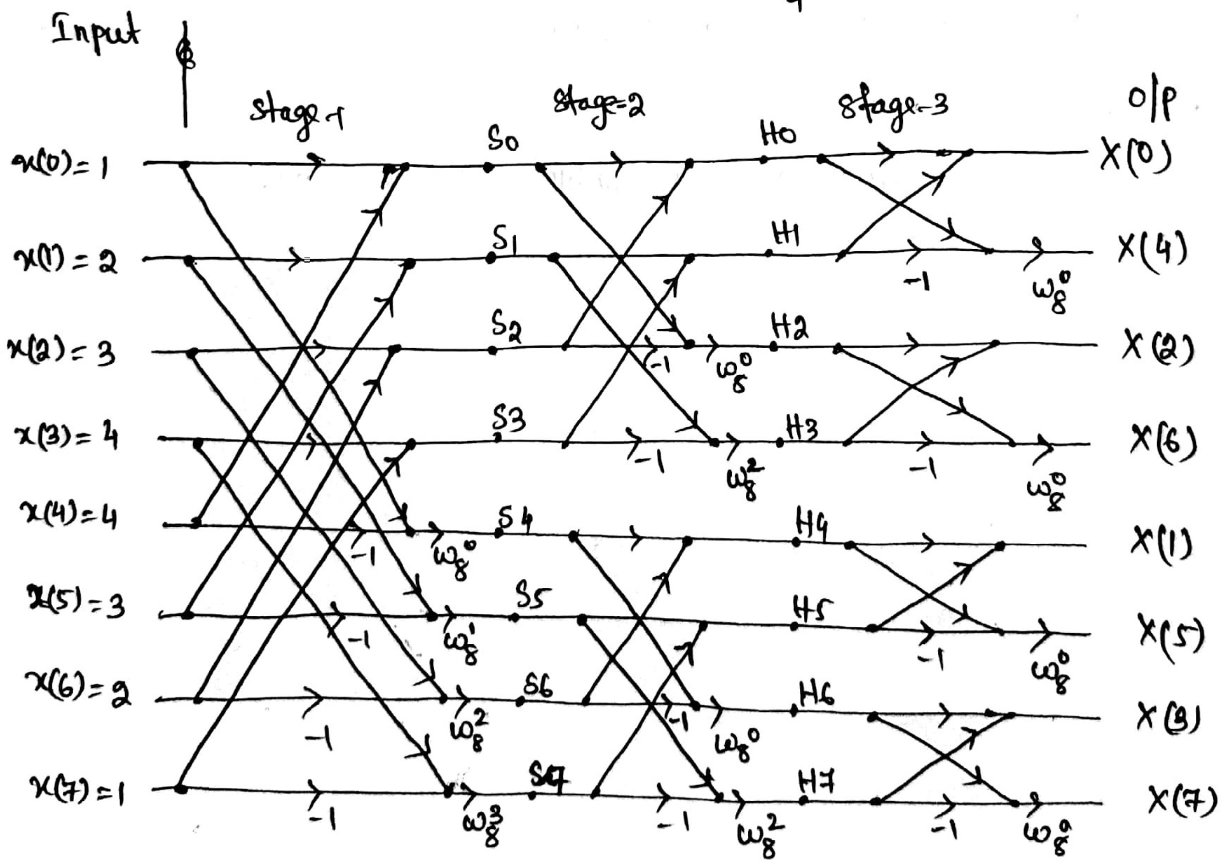
$W_8^0 = e^{-j\left(\frac{2\pi}{8}\right)0} = e^0 = 1$

$W_8^1 = e^{-j\left(\frac{2\pi}{8}\right)1} = e^{-j\pi/4} = \cos\pi/4 - j\sin\pi/4 = 0.707 - j0.707$

$= \cancel{0.707 - j0.707}$

$W_8^2 = e^{-j\left(\frac{2\pi}{8}\right)2} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = -j$

$W_8^3 = e^{-j\left(\frac{2\pi}{8}\right)3} = e^{-j3\pi/4} = \cos\frac{3\pi}{4} - j\sin 3\pi/4 = -0.707 - j0.707$



I/P	O/P of stage-1	O/P of stage-2	O/P
$x(0) = 1$	$S_0 = x(0)+x(4) = 1+4 = 5$	$H_0 = S_0+S_2 = 5+5 = 10$	$X(0) = H_0 + H_1$ $= 10+10 = 20$
$x(1) = 2$	$S_1 = x(1)+x(3) = 2+3 = 5$	$H_1 = S_1+S_3 = 5+5 = 10$	$X(4) = [H_0 - H_1]\, W_8^0$ $= [10-10]\, 1 = 0$
$x(2) = 3$	$S_2 = x(2)+x(6) = 3+2 = 5$	$H_2 = (S_0 - S_2) W_8^0 = 5-5 = 0$	$X(2) = H_2 + H_3 = 0+0 = 0$
$x(3) = 4$	$S_3 = x(3)+x(7) = 4+1 = 5$	$H_3 = (S_1 - S_3)\, W_8^2$ $= (5-5)(-j) = 0$	$X(6) = [H_2 - H_3]\, W_8^0$ $= [0-0]\, 1 = 0$
$x(4) = 4$	$S_4 = [x(0) - x(4)]\, W_8^0 = -3$	$H_4 = S_4 + S_6 = -3-j$	$X(1) = H_4 + H_5$ $= -3-j-2.828-j1.414$ $= -5.828-j2.414$
$x(5) = 3$ ~~$x(6)=2$~~	$S_5 = [x(1) - x(5)]\, W_8^1$ $= (2-3)(0.707-j0.707)$ $= -0.707+j0.707$	$H_5 = S_5 + S_7$ $= -0.707+j0.707$ $-2.121-j2.121$ $= -2.828-j1.414$	
$x(6) = 2$	$S_6 = [x(2) - x(6)]\, W_8^2$ $= (3-2)(-j) = -j$	$H_6 = (S_4 - S_6)\, W_8^0$ $= -3+j$	$X(5) = [H_4 - H_5]\, W_8^0$ $= (-3-j+2.828+j1.414)\, 1$ $= -0.172-j0.414$
$x(7) = 1$	$S_7 = [x(3) - x(7)]\, W_8^3$ $= (4-1)(-0.707-j0.707)$ $= -2.121-j2.121$	$H_7 = (S_5 - S_7)\, W_8^2$ $= (-0.707+j0.707+$ $2.121+j2.121)(-j)$ $= 2.828-j1.414$	$X(3) = H_6 + H_7 = -3+j+2.828$ $-j1.414$ $= -0.172-j0.414$ $X(7) = [H_6 - H_7]\, W_8^0$ $= (-3+j-2.828+j1.414)\, 1$ $= -5.828+j2.414$

$X(K) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$ Ans

~~$X(K) = 20, 0, 0, 0$~~

$X(K) = \{20, -5.828-j2.414, 0, -0.172-j0.414, 0, -0.172-j0.414, 0, -5.828+j2.414\}$

Ans

Q2// Compute 4-point DFT of the following sequences using DIF algorithm

$$x(n) = \{0, 1, 2, 3\}$$

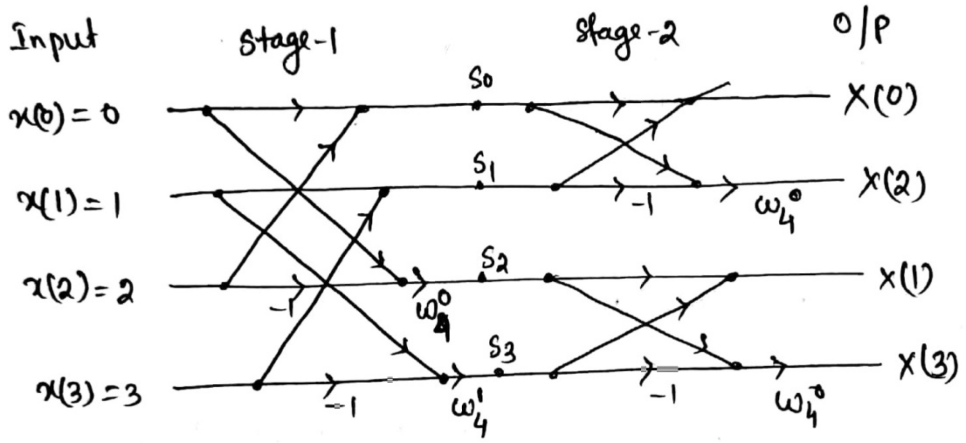
Sol^n

Here $N = 4$

We know $W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$ $\left[W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k} = \text{twiddle factor}\right]$

$$W_4^0 = e^{-j\left(\frac{2\pi}{4}\right)0} = e^0 = 1$$

$$W_4^1 = e^{-j\left(\frac{2\pi}{4}\right)1} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = -j$$



Input	O/p of stage-1	O/p.
$x(0) = 0$ $x(1) = 1$ $x(2) = 2$ $x(3) = 3$	$S0 = x(0) + x(2)$ $= 0 + 2 = 2$ $S1 = x(1) + x(3)$ $= 1 + 2 = 3$ $S2 = [x(0) - x(2)]\, W_4^0$ $= [0 - 2]\, 1 = -2$ $S3 = [x(1) - x(3)]\, W_4^1$ $= [1 - 2](-j)$ $= j$	$X(0) = S0 + S1$ $= 2 + 3 = 5.$ $X(2) = [S0 - S1]\, W_4^0$ $= [2 - 3]\, 1 = -1$ $X(1) = S2 + S3$ $= -2 + j$ $X(3) = [S2 - S3]\, W_4^0$ $= [-2 - j]\, 1$ $= -2 - j$

$$X(k) = \{X(0), X(1), X(2), X(3)\} = \{5, -2+j, -1, -2-j\}$$ Ans

Differcance ~~and~~ between DIT & DIF Algorithm :→

① For decimation-in-time (DIT), the i/p is bit reversed while the o/p is in natural order. whereas for decimation-in-frequency the i/p is in natural order while the o/p is bit reversed order.

② The DIF butterfly is slightly different from the DIT wherein DIF the complex multiplication takes place after the add-subtract operation.

Similarities betⁿ DIT & DIF Algorithm :→

Both algorithms require $N\log_2 N$ operations to compute the DFT. Both algorithm can be done in-place & both need to perform bit reversal at some place during the computation.

Introduction to digital filters (FIR filters)

Introduction →

Filter :→ Filtering is a process by which the frequency spectrum of a signal can be modified, reshaped or manipulated to achieve some desired objectives.

These objectives can be listed as under :

i) To eliminate noise which may be contaminated in a signal.

ii) To remove signal distoration which may be due to imperfection transmission channel.

iii) To separate two or more distinct signals which are purposely mixed for maximising channel utilization.

iv) To resolve signals into their frequency components.

v) To demodulate the signals which were modulated at the transmitter end.

vi) To convert digital signal into analog signals.

vii) To limit the bandwidth of signals.

Types of Filters :→

Filters are basically of two types depending upon the type of signal to be processed.

(i) Analog filters
(ii) Digital filters

Analog filters :→

Analog filter may be defined as, a system in which both i/p & the o/p are continuous-time signals.

(Block diagram of Analog filter)

Digital filters :→

Digital filters may be defined as a system in which both the i/p & o/p are discrete-time- ~~sig~~ signals.

(Block diagram of Digital filter)

Difference bet^n Digital filters & Analog filters.

Digital Filter	Analog Filter
1. A digital filter processes & generates digital data.	1. Analog filter processes analog i/ps & generates analog o/ps.
2. A digital filter consists of elements like adder, multiplier & delay unit.	2. Analog filters are constructed from active or passive electronics components.
3. Digital filters is described by a difference equation.	3. Analog filter is described by a differential equation.
4. The frequency response can be changed by changing the filter co-efficients.	4. The frequency response of an analog filter can be modified by changing the components.

~~Advantages of~~ Advantages And Disadvantages of Digital filters

Advantages

1. Unlike analog filter, the digital filter performance is not influenced by component ageing, temperature & power supply variations.
2. A digital filter is highly immune to noise & ~~posses considera~~ & possess ~~param~~ considerable parameter stability.
3. Digital filters can be operated over a wide range of frequencies.
4. Multiple filtering is possible only in digital filter.
5. Digital filters afford a wide variety of shapes for the amplitude & phase response.
6. The co-efficients of digital filters can be programmed or altered any time to obtain the desired characteristics.

Disadvantage :-

1. The quantization error ~~consists arises~~ arises due to finite word length in the representation of signals & parameters

Types of Discrete time Systems | Digital filter :→

A discrete-time system can be realized in two ways :-

1) Recursive
2) Non-recursive

Recursive

For recursive realization, the current o/p y(n) is a function of past o/ps, past & present i/ps. This form corresponds to an Infinite Impulse Response (IIR) system or IIR filters.

Non-Recursive :-

For non-recursive realization, the current o/ps y(n) is a function of past & present i/ps only. This form corresponds to finite impulse response (FIR) system or FIR filter.

Advantage of FIR filter over IIR filter

1. FIR filters are always stable.
2. FIR filters with exactly linear phase can easily be designed.
3. FIR filters can be realized in both recursive & non-recursive structures.
4. Excellent design methods are available for various kinds of FIR filters.

Disadvantage of FIR filter :→

1. FIR filters are very costly, as it requires considerably more arithmetic operations & hardware components such as multipliers, adders & delay elements.
2. Memory requirement & execution time are very high.

Differences betn FIR & IIR filters

FIR filter	IIR filter
1. These filters can be easily designed to have perfectly linear phase.	1. These filters do not have linear phase.
2. FIR filters can be realized recursively & non-recursively.	2. IIR filters are easily realized recursively.
3. Greater flexibility to control the shape of their magnitude response.	3. Less flexibility, usually limited to specific kind of filters.
4. Errors due to round off noise are less severe in FIR filters, mainly because feedback is not used.	4. The round off noise in IIR filters are more.

Chapter-4

4. Discuss Fourcier Transform : Its Application Properties

4.1 Discuss Fourier Transform :→

Introduction :→

i) Frequency analysis of discrete-time signals is usually performed on a digital signal processor which may be a simple digital computer or specially designed digital hardware.

ii) To perform the frequency analysis on a discrete-time signal $x(n)$ it is required to first convert the time-domain sequence to an equivalent frequency domain representation. Such a frequency domain representation is obtained by the fourier transform $X(\omega)$ of the sequence $x(n)$

iii) $X(\omega)$ is a continuous function of frequency & therefore, the fourier transform is not a computational convenient representation of $x(n)$.

iv) In this segment the sequence $x(n)$ is represented by the samples of its spectrum $X(\omega)$ i-e, the continuous frequency domain representation $X(\omega)$ is converted to a discrete-frequency domain representation. Such a frequency domain representation is called Discrete Fourier transform.

4.2 Determine frequency domain sampling & reconstruction of discrete time signal :→

Let us consider an a periodic discrete-time ~~signal~~ sequence $x(n)$ ~~having~~ with fourier transform

$$\boxed{X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}}$$

~~Let~~ Suppose we sample $X(\omega)$ periodically in a frequency at a spacing of $\Delta\omega$ radians between the successive samples. As we know that $X(\omega)$ is periodic with period 2π, we requires the samples only in fundamental frequency range. We take N equi-distance samples in the interval of $0 \le \omega \le 2\pi$ with

Spacing $\Delta\omega = \frac{2\pi}{N}$ which is shown in fig 1.1

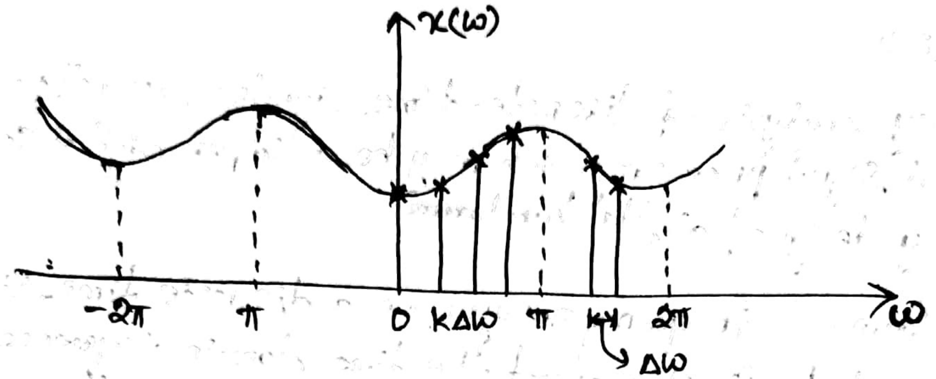


Fig 1.1. (Frequency domain sampling of the fourier Transform)

Let us evaluate eqn (1.1) at $\omega = \frac{2\pi k}{N}$, we have

$$X\left(\frac{2\pi K}{N}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi K}{N}n} \qquad K = 0, 1, 2, \dots N-1 \quad \text{---(1.2)}$$

The eqn (1.2) further can be subdivided into an infinite no. of summations, where each sum contain N terms. Thus

$$X\left(\frac{2\pi K}{N}\right) = \dots + \sum_{n=-N}^{-1} x(n) e^{-j\frac{2\pi K}{N}n} + \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi K}{N}n} + \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi K}{N}n} + \dots$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=mN}^{mN+N-1} x(n) e^{-j\frac{2\pi K}{N}n} \quad \text{———(1.3)}$$

By changing the index in the inner summation from n to n−mN, we obtain (i.e, n = n−mN ⟹ mN = 0)

$$X\left(\frac{2\pi K}{N}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-mN) e^{-j\frac{2\pi K}{N}(n-mN)}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-mN) e^{-j\frac{2\pi K}{N}n} \quad \left(\text{Since } e^{j2\pi Km} = 1\right)$$

———(1.4)

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If we interchange the order of the summation, we obtain

$$X\left(\frac{2\pi K}{N}\right)n = X(k) = \sum_{n=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x(n-mN)e^{-j\frac{2\pi k}{N}n}\right] \quad (1.5)$$

where $X\left(\frac{2\pi K}{N}n\right)$ is replaced by $X(k)$ for simplicity.

We defined the signal $x_p(n) = \sum_{m=-\infty}^{\infty} x(n-mN)$ ——— (1.6)

~~We defined the signal~~ $\cancel{x_p(n) = \sum_{m=-\infty}^{\infty} x\,n-mN}$

which is obtained by periodic repetition of $x(n)$ every N samples.
The signal $x_p(n)$ is clearly periodic with fundamental period N.
Therefore the signal $x_p(n)$ can also be expended in a Fourier Series as

$$x_p(n) = \sum_{K=0}^{N-1} C_K e^{j\frac{2\pi k}{N}n} \qquad n = 0, 1, 2 \cdots N-1 \quad (1.7)$$

where C_K are the co-efficients in series representation.

$$\cancel{C_K =} x_p(n) = \frac{1}{N}\sum_{K=0}^{N-1} C_K e^{-j\frac{2\pi K}{N}n} \quad , n = 0, 1, 2 \cdots N-1 \quad (1.8)$$

~~Where the division by 1/N~~

~~By comparing eqn~~

As we have C_K already obtained;

$$C_K = \sum_{n=0}^{N-1} x_p(n)e^{-j\frac{2\pi K}{N}n}, \quad K = 0, 1, 2, \cdots N-1 \quad (1.9)$$

Comparing eqn (1.5) & (1.9), we obtained

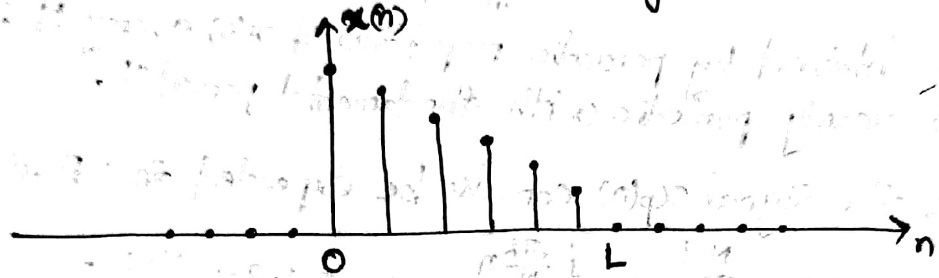
$$C_K = X\left(\frac{2\pi K}{N}n\right) = X(k) \quad , K = 0, 1, 2, \cdots N-1 \quad (1.10)$$

Therefore $$\boxed{x_p(n) = \frac{1}{N}\sum_{K=0}^{N-1} X\left(\frac{2\pi}{N}K\right)e^{j2\pi kn/N}} , n = 0, 1, 2, \cdots N-1 \quad (1.11)$$

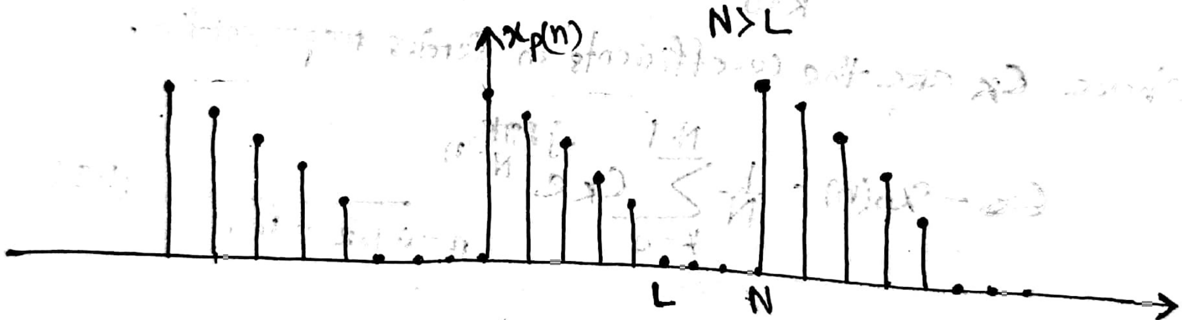
From we obtain the periodic signal $x_p(n)$ from the samples of $X(\omega)$. However, this does not imply that we can recover $x(n)$ from the samples. To get this we have to find a rel^n bet^n $x_p(n)$ & $x(n)$.

As $x_p(n)$ is the periodic extension of $x(n)$ given by (1.7), $x(n)$ can be recovered from $x_p(n)$ if there is no aliasing in the time domain that is $x(n)$ is time-limited to less than the period (N) of $x_p(n)$.

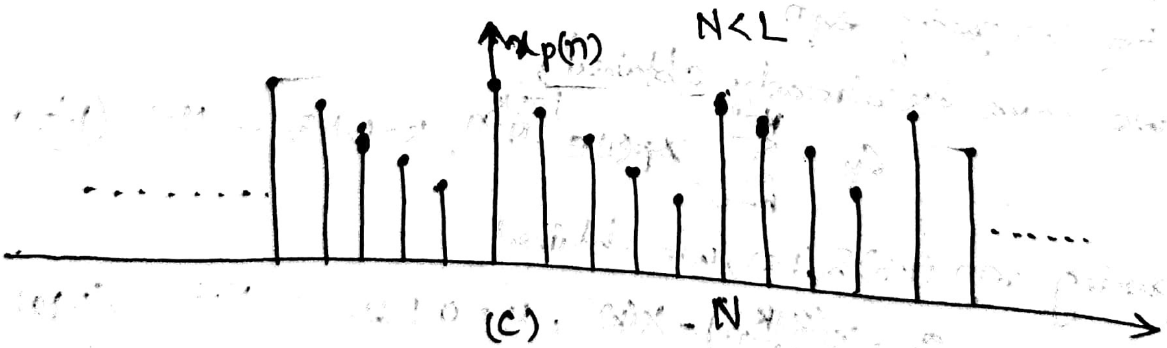
Let consider a finite-duration sequence $x(n)$ which is defined as $x(n) \neq 0$, $0 \leq n \leq L-1$ as shown in fig (2.1 (a))



(a)



(b)



(c)

Fig 2.1. A periodic sequence $x(n)$ of length L. & its periodic extension for $N \geq L$ (no aliasing) & $N < L$ (aliasing)

From the above figure we observe that for $N \geq L$

$$x(n) = x_p(n), \quad 0 \leq n \leq N-1$$

So $x(n)$ can be recovered from $x_p(n)$ as shown in fig (2.1b) on the other hand of $N<L$, $x(n)$ can't be recovered from $x_p(n)$ due to time domain aliasing as shown in fig (2.1c).

Thus the spectrum of an aperiodic discrete time signal with finite duration L, can be recovered from its samples of frequencies

$$\boxed{\omega_K = \frac{2\pi k}{N} \text{ if } N \geq L}$$

$$\text{So, } x(n) = \begin{cases} x_p(n), & 0 \leq n \leq N-1 \\ 0, & \text{elsewhere} \end{cases} \qquad \text{--- (1.12)}$$

As $x(n) = x_p(n)$ for $0 \leq n \leq N-1$ then from eq^n (1.11), we obtain

$$\boxed{x(n) = \frac{1}{N}\sum_{n=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{\frac{j2\pi kn}{N}}, \quad 0 \leq n \leq N-1} \qquad \text{--- (1.13)}$$

4.3 State & explain Discrete-fourier-Transform :-

The discrete-time fourier transform (DTFT) or simply the fourier transform of a discrete-time-sequence $x(n)$ is defined by

$$\boxed{X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}} \qquad \text{--- (1.14)}$$

This eqn represents the fourier series representation of the periodic function $X(\omega)$. Hence the fourier co-efficients $x(n)$ can be determined from $X(\omega)$ using Fourier integral expressed by

$$\boxed{x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega} \qquad \text{--- (1.15)}$$

called the inverse discrete-time Fourier transform (IDTFT)

4.4 State & explain Discrete fourier transform.

i) The DFT is a powerful computation tool which allows us to evaluate the fourier transform ($X e^{j\omega}$) on a digital computer.

ii) DFT is obtained by sampling one period of the fourier transform at a finite no. of frequency points.

iii) DFT plays an important role in the implementation of many signal processing algorithm.

iv) DFT is used to perform Linear filtering operations in the frequency domain.

The DFT of a sequence ~~x(n)~~ $x(n)$ of length N is given by

$$\boxed{X(K) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi Kn}{N}} \quad ; K = 0, 1, \cdots N-1}$$

where $x(n)$ = discrete time finite sequence

$X(K)$ = N-point DFT sequence

N = length of the sequence

K = discrete frequencies where the samples taken

IDFT

The process to recover the sequence $x(n)$ from the frequency samples is called inverse DFT (IDFT), which is given by

$$\boxed{x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi Kn}{N}} \quad , n = 0, 1, \cdots N-1}$$

Problems :-

Q1 Find the 4-point DFT of the sequence $x(n) = \{1, 2, 1, 1\}$

Soln The N-point DFT of $x(n)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}, \quad k = 0, 1, 2 \cdots N-1$$

Here $N = 4$

$$\text{Hence } X(k) = \sum_{n=0}^{3} x(n) e^{\frac{-j2\pi kn}{N}}, \quad k = 0, 1, 2, 3$$

For $K = 0$

$$X(0) = \sum_{n=0}^{3} x(n) e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{3} x(n) \qquad (\because e^{0} = 1)$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 2 + 1 + 1 = 5$$

For $K = 1$

$$X(1) = \sum_{n=0}^{3} x(n) e^{\frac{-j2\pi n}{4}} = \sum_{n=0}^{3} x(n) e^{\frac{-j\pi n}{2}}$$

$$= x(0)e^{0} + x(1) e^{-j\pi/2} + x(2) e^{-j\pi} + x(3) e^{-j3\pi/4}$$

$$= 1 + 2(\cos \pi/2 - j\sin \pi/2) + 1 \cdot x(\cos\pi - j\sin\pi) + 1 \times \left(\cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}\right)$$

$$= 1 + 2 \times (-j) + (-1) + j = 1 - 2j - 1 + j = -j$$

For $k = 2$

$$X(2) = \sum_{n=0}^{3} x(n) e^{\frac{-j4\pi n}{4}}$$

$$= \sum_{n=0}^{3} x(n) e^{-j\pi n}$$

$$= \cancel{x_0}\; x(0) e^{0} + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$

$$= 1 + 2 \times (-1) + 1 \times 1 + 1 \times (-1) = 1 - 2 + 1 + 1 = 1$$

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For $K=3$

$$X(3) = \sum_{n=0}^{3} x(n) e^{\frac{-j6\pi n}{4}}$$

$$= \sum_{n=0}^{3} x(n) e^{\frac{-j3\pi n}{2}}$$

$$= x(0) + x(1) e^{\frac{-j3\pi}{2}} + x(2) e^{\frac{-j6\pi}{2}} + x(3) e^{\frac{-j9\pi}{2}}$$

$$= 1 + 2 \times j + 1 \times (-1) + 1 \times (-j)$$

$$= 1 + 2j - 1 - j = j$$

Hence, $X(k) = \{5, -j, 1, j\}$ Ans.

Q. Find the N-point DFT for $x(n) = a^n$ for $0 < a < 1$

Soln The N-point DFT is

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}, \quad k = 0, 1, \cdots N-1$$

$$= \sum_{n=0}^{N-1} a^n e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} \left(a e^{\frac{-j2\pi k}{N}}\right)^n$$

$$= \frac{1 - \left(a e^{\frac{-j2\pi k}{N}}\right)^N}{1 - a e^{\frac{-j2\pi k}{N}}} \qquad \left[\because \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}\right]$$

$$= \frac{1 - a^N e^{-j\frac{2\pi kN}{N}}}{1 - a e^{\frac{-j2\pi k}{N}}}$$

$$= \frac{1 - a^N e^{-j2\pi k}}{1 - a e^{\frac{-j2\pi k}{N}}}$$

$$= \frac{1 - a^N}{1 - a e^{\frac{-j2\pi k}{N}}}$$

where $K = 0, 1 \cdots N-1$ $\qquad \left(\because e^{-j2\pi k} = 1 \text{ for } K = 0, 1 \cdots N-1\right)$

Q3// Find the 4-point DFT of the sequence $x(n) = \cos\frac{\pi n}{4}$

Sol^n

Given $N = 4$

So $x(n) = \{\cos 0, \cos(\frac{\pi}{4}), \cos(\frac{2\pi}{4}), \cos(\frac{3\pi}{4})\}$

$= \{1, 0.707, 0, -0.707\}$

$$X(k) = \sum_{n=0}^{3} x(n) e^{\frac{-j2\pi kn}{4}}, \quad k = 0,1,2,3$$

For $K=0$

$$X(0) = \sum_{n=0}^{3} x(n) \qquad (\because e^{0} = 1, \text{ as } k=0)$$

$= x(0) + x(1) + x(2) + x(3)$

$= 1 + 0.707 + 0 - 0.707 = 1$

For $K = 1$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi n}{4}} \qquad (\because k=1)$$

$= x(0) + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-j\frac{3\pi}{2}}$

$= 1 + 0.707 \times (-j) + 0 \times (-1) + (-0.707) \times j$

$= 1 - j0.707 - j0.707$

$= 1 - j1.414$

For $K = 2$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n} \qquad (\because k=2)$$

$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$

$= 1 + 0.707 \times (-1) + 0 \times (1) + (-0.707) \times (-1)$

$= 1 - 0.707 + 0.707$

$= 1$

For $k=3$

$$X(2) = \sum_{n=0}^{3} x(n) e^{\frac{-j\pi n 3}{2}} \quad (\because k=3)$$

$$= \cancel{x(0) + x(1) e^{\frac{-j\pi}{2}} + x(2) e^{-j\pi} + x(3) e^{-j}}$$

$$= x(0) + x(1) e^{-j\frac{3\pi}{2}} + x(2) e^{-j3\pi} + x(3) e^{-j\frac{9\pi}{2}}$$

$$= 1 + 0.707 \times j + 0 \times (-1) + (-0.707) \times (-j)$$

$$= 1 + j0.707 + j0.707$$

$$= 1 + j1.414$$

Hence, $X(k) = \{1, 1-j1.414, 1, 1+j1.414\}$ Ans

Q4 Find the IDFT of $X(k) = \{3, 2+j, 1, 2-j\}$

Soln The IDFT is

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}} \quad , n = 0, 1, \ldots N-1$$

Here $N = 4$,

$$\text{so } x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{\frac{j2\pi kn}{N}} \quad , n = 0, 1, 2, 3$$

For $n=0$

$$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{0} \quad (\because e^{0} = 1)$$

$$= \frac{1}{4}\left[X(0) + X(1) + X(2) + X(3)\right]$$

$$= \frac{1}{4}\left[3 + 2 + j + 1 + 2 - j\right]$$

$$= \frac{1}{4} \times 8 = 2$$

For $n=1$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{\frac{j2\pi k}{4}} \quad (\because n=1)$$

$$= \frac{1}{4}\left[X(0) + X(1)e^{\frac{j\pi}{2}} + X(2)e^{j\pi} + X(3)e^{\frac{j3\pi}{2}}\right]$$

$$= \frac{1}{4}\left[3+(2+j)\times j + 1\times(-1) + (2-j)\times(-j)\right]$$

$$= \frac{1}{4}\left[3+2j-1-1-2j-1\right]$$

$$= \frac{1}{4}\times 0 = 0$$

For $n=2$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{\frac{j4\pi k}{4}} \quad (\because n=2)$$

$$= \frac{1}{4}\left[X(0) + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right]$$

$$= \frac{1}{4}\left[3+(2+j)(-1) + 1\times 1 + (2-j)\times(-1)\right]$$

$$= \frac{1}{4}\left[3-2-j+1-2+j\right]$$

$$= \frac{1}{4}\times 0 = 0$$

For $n=3$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{\frac{j6\pi k}{4}} \quad (\because n=3)$$

$$= \frac{1}{4}\left[X(0) + X(1)e^{\frac{j3\pi}{2}} + X(2)e^{j3\pi} + X(3)e^{\frac{j9\pi}{2}}\right]$$

$$= \frac{1}{4}\left[3+(2+j)(-j) + 1\times(-1) + (2-j)\times j\right]$$

$$= \frac{1}{4}\left[3-2j+1-1+2j+1\right]$$

$$= \frac{1}{4}\times 4 = 1$$

Hence, $x(n) = \{2, 0, 0, 1\}$ Ans

4.5 Compute DFT as Linear Transformation :→

The expressions for DFT & IDFT are given as

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad k = 0, 1, \cdots N-1 \quad \text{—(1)}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk}, \quad n = 0, 1, \cdots N-1 \quad \text{—(2)}$$

where $W_N = e^{\frac{-j2\pi}{N}}$

The computation of each point of the DFT can be accomplished by N complex multiplication & (N-1) complex additions. So the N-point DFT values can be computed in total N^2 complex multiplication & N(N-1) complex additions.

The DFT & IDFT can be viewed as linear transformations on sequences $\{x(n)\}$ & $\{X(k)\}$

From eqn (1)

$$X(0) = x(0)W_N^0 + x(1)W_N^0 + x(2)W_N^0 + \cdots + x(N-1)W_N^0$$

$$X(1) = x(0)W_N^0 + x(1)W_N^1 + x(2)W_N^2 + \cdots + x(N-1)W_N^{N-1}$$

$$X(2) = x(0)W_N^0 + x(1)W_N^2 + x(2)W_N^4 + \cdots + x(N-1)W_N^{2N-1}$$

$$\vdots$$

$$X(N-1) = x(0)W_N^0 + x(1)W_N^{N-1} + \cdots + x(N-1)W_N^{(N-1)(N-1)}$$

The above eqn can be represented in terms of matrices as

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \begin{bmatrix} W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & \cdots & W_N^N \\ \vdots & & & \\ W_N^0 & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$\Rightarrow \boxed{X_N = x_N W_N}$$

where X_N = N-point vector of frequency samples

x_N = N-point vector of the signal sequences $x(n)$

$W_N = N \times N$ matrix = matrix of linear transformation

The N-point DFT may be expressed as

$$X_N = W_N x_N$$

& N-point IDFT may be given as

$$x_N = W_N^{-1} X_N$$

$$= \frac{1}{N} W_N^{*} X_N$$

W_N^{*} is complex conjugate of W_N

$$W_N^{-1} = \frac{1}{N} W_N^{*}$$

$$\Rightarrow W_N \cdot W_N^{*} = N I_N$$

where I_N is the $N \times N$ identity matrix.

4.6 Relate DFT to Other Transform

(1) Relationship to the Fourier Transform

The fourier transform $X(e^{j\omega})$ of a finite duration sequence $x(n)$ having length N is given by

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \quad -(1)$$

where $X(e^{j\omega})$ is a continuous function of ω.

The discrete fourier transform of $x(n)$ is given by

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi K n}{N}} \qquad K = 0, 1, \cdots N-1 \quad -(2)$$

Comparing the eq^n (1) & (2), we find that the DFT of $x(n)$ is sampled version of the fourier transform of the sequence & is given by

$$X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \qquad k = 0, 1, \cdots N-1 \quad \text{--- (3)}$$

(ii) <u>Relation to the z-transform</u>

Let $x(n)$ is a discrete-time sequence. The z-transform of $x(n)$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{--- (1)}$$

If $x(n)$ is a finite duration sequence of length N, then

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} \quad \text{--- (2)}$$

From IDFT, $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}}$ --- (3)

Putting the value of (3) in (2), we got

$$X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}} \right] z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} e^{\frac{j2\pi kn}{N}} z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{\frac{j2\pi k}{N}} z^{-1} \right)^n$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \left[\frac{1 - \left(e^{\frac{j2\pi k}{N}} z^{-1}\right)^N}{1 - e^{\frac{j2\pi k}{N}} z^{-1}} \right] \qquad \left[\because \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \right]$$

$$\Rightarrow X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \left[\frac{1 - e^{j2\pi k} z^{-N}}{1 - e^{\frac{j2\pi k}{N}} z^{-1}} \right]$$

$$\Rightarrow \boxed{X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{\frac{j2\pi k}{N}} z^{-1}}} \qquad \because e^{j2\pi k} = 1 \text{ for all values of } k$$

4.7 Discuss the Property of DFT.

Properties of DFT :→

① Periodicity :→

$$\text{If } x(n) \xleftrightarrow[N]{DFT} X(k)$$

then $x(n+N) = x(n)$

$$X(k+N) = X(k)$$

Proof :→ $$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi(k+N)}{N}n}$$

$$= \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}} e^{-j2\pi n}$$

$$= \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}} = X(k) \quad \text{(Proved)}$$

② Linearity Property :→

$$\text{If } x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$$

$$\text{and } x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$$

then for any real valued or Complex valued constants a_1 & a_2

$$a_1x_1(n) + a_2x_2(n) \xleftrightarrow[N]{DFT} a_1X_1(k) + a_2X_2(k)$$

Proof $$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}}$$

Let $x(n) = a_1x_1(n) + a_2x_2(n)$

So, $$X(k) = \sum_{n=0}^{N-1} \{a_1x_1(n) + a_2x_2(n)\} e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} a_1 x_1(n) e^{-j\frac{2\pi nk}{N}} + \sum_{n=0}^{N-1} a_2 x_2(n) e^{-j\frac{2\pi nk}{N}}$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi nk}{N}} + a_2 \sum_{n=0}^{N-1} x_2(n) e^{-j\frac{2\pi nk}{N}}$$

$$\boxed{X(k) = a_1 X_1(k) + a_2 X_2(k)}$$ Proved

Additional DFT Properties:→

1. Time reversal Property

$$\text{If } x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$\text{then } x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N = X(N-k)$$

2. Circular Time Shifting Property :→

$$\text{If } x(n) \xleftrightarrow{DFT} X(k)$$

$$\text{then } x((n-l))_N \xleftrightarrow{DFT} X(k).e^{-j2\pi kl/N}$$

3. Circularly frequency shifting Properties :→

$$\text{If } x(n) \xleftrightarrow{DFT} X(k)$$

$$\text{then } x(n).e^{\frac{j2\pi kl}{N}} \xleftrightarrow{DFT} X((k-l))_N$$

4. Complex Conjugate Properties :→

$$\text{If } x(n) \longleftrightarrow X(k)$$

$$x^*(n) \longleftrightarrow X^*((-k))_N = X^*(N-k)$$

~~5. Circularly Correlation~~

5. Multiplication of two Sequence :→

$$\text{If } x(n) \xleftrightarrow{DFT} X(k)$$

$$y(n) \xleftrightarrow{DFT} Y(k)$$

$$\text{then } x(n).y(n) \xleftrightarrow{DFT} \frac{1}{N}\left[X(k) \, N \, Y(k)\right]$$

6. Parseval's Theorem Property:→

$$\text{If } x(n) \longleftrightarrow X(k)$$
$$y(n) \longleftrightarrow Y(k)$$

$$\text{then } \sum_{n=0}^{N-1} x(n) \cdot y^{*}(n) \longleftrightarrow \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot Y^{*}(k)$$

~~Multiplication of~~

4.8. Multiplication of Two DFTs & Circular Convolution :→

Let us assume that we have two finite duration sequence of length N.

Their N-point DFTs are

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad k = 0,1,2 \cdots N-1$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad k = 0,1,2 \cdots N-1$$

Let multiply these two DFTs $X_1(k)$ & $X_2(k)$ which results another DFT $X_3(k)$ defined as

$$X_3(k) = X_1(k)\, X_2(k) \quad , \quad k = 0,1,2 \cdots N-1$$

Proof Let $X_3(k) = \text{DFT}\left[x_3(m)\right]$

$$x_3(m) \xleftrightarrow[N]{DFT} X_3(k).$$

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k)\, e^{\frac{j2\pi km}{N}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[X_1(k)\, X_2(k)\right] e^{\frac{j2\pi km}{N}}$$

Putting the value of $X_1(k)$ & $X_2(k)$

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n) e^{-\frac{j2\pi kn}{N}}\right]\left[\sum_{l=0}^{N-1} x_2(l) e^{-\frac{j2\pi kl}{N}}\right] e^{\frac{j2\pi km}{N}}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n-l)}{N}}\right]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1}\left\{e^{\frac{j2\pi(m-n-l)}{N}}\right\}^{k}\right]$$

Let $a = e^{\frac{j2\pi(m-n-l)}{N}}$

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a=1 \\ \frac{1-a^N}{1-a}, & a \neq 1 \end{cases}$$

we observe that $a = e^{\frac{j2\pi(m-n-l)}{N}} = 1$

$$= \sum_{k=0}^{N-1} 1 = N \qquad \left(\because m-n-l = \overset{\text{is an integer}}{PN}\right)$$

If in the above expression we are substituting $m-n-l = PN$ then $\sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n-l)}{N}} = N$

$$\Rightarrow x_3(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)\cdot N$$

$$= \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(m-n-PN)$$

$$\boxed{x_3(m) = \sum_{n=0}^{N-1} x_1(n)\cdot x_2((m-n))_N} \quad m = 0,1,\cdots N-1$$

which is known as Circular Convolution, where $0 \le m \le N-1$.

Thus multiplication of the DFT of 2 sequence is equivalent to the Circular Convolution of the 2 sequences in the time domain.

$$\boxed{X_3(m) = x_1(n) \circledN x_2 N}$$

Methods for Circular Convolution :→

The methods that are used to find the circular convolution of two sequences are

1. Concentric Circle method
2. Matrix multiplication method.

Concentric Circle Method :→

Given two sequences $x_1(n)$ & $x_2(n)$ the circular convolution of these two sequences $x_3(n) = x_1(n) \circledN x_2(n)$ can be found by using the following steps.

1. Graph N samples of $x_1(n)$ as equally spaced points around an outer circle in counterclockwise direction.
2. Start at the same point as $x_1(n)$ graph N samples of $x_2(n)$ as equally spaced points around an inner circle in clockwise direction.
3. Multiply corresponding samples on the two circles & sum the products to produce o/p.
4. Rotate the inner circle one sample at a time in counterclockwise direction & go to step 3 to obtain the next value of o/p.
5. Repeat step No-4 until the inner circle first sample lines up with the first sample of the exterior circle once again.

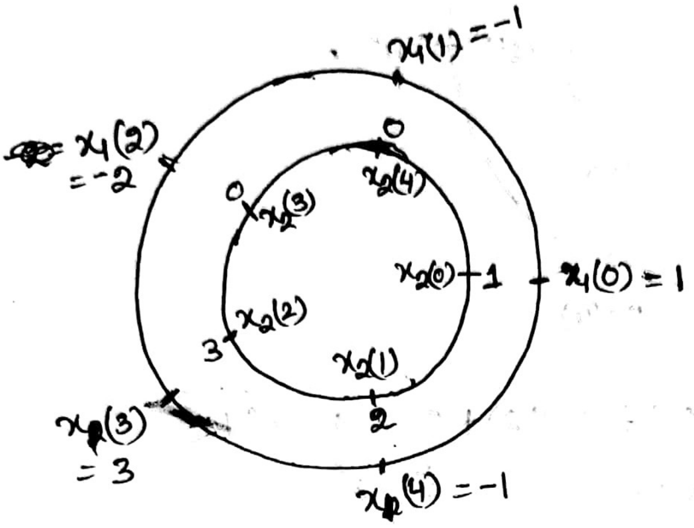
Problems

Q1 Find the circular convolution of two finite duration sequences $x_1(n) = \{1, -1, -2, 3, -1\}$, $x_2(n) = \{1, 2, 3\}$.

Soln [N.B → To find circular convolution, both sequences must be of same length. Therefore we add two zeros to the sequence $x_2(n)$ & use concentric circle method to find circular convolution.]

$$x_1(n) = \{1, -1, -2, 3, -1\}$$
$$x_2(n) = \{1, 2, 3, 0, 0\}$$

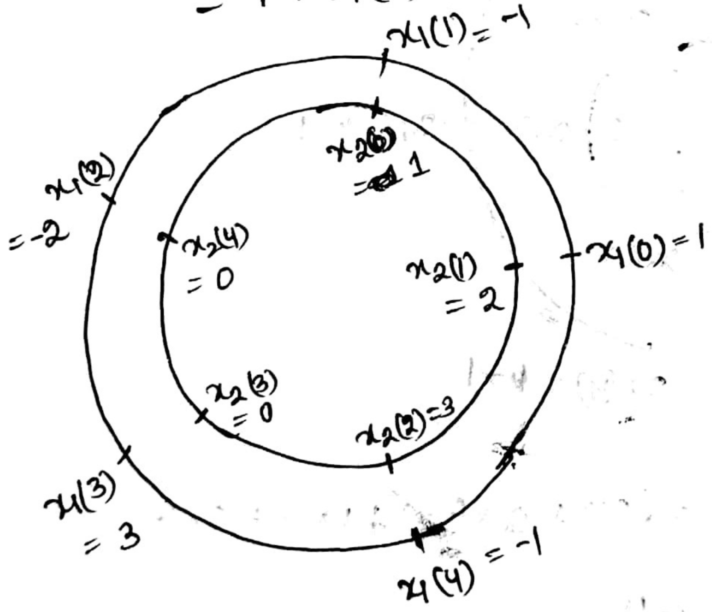
Step 1



Multiply corresponding samples on the circle & add to obtain

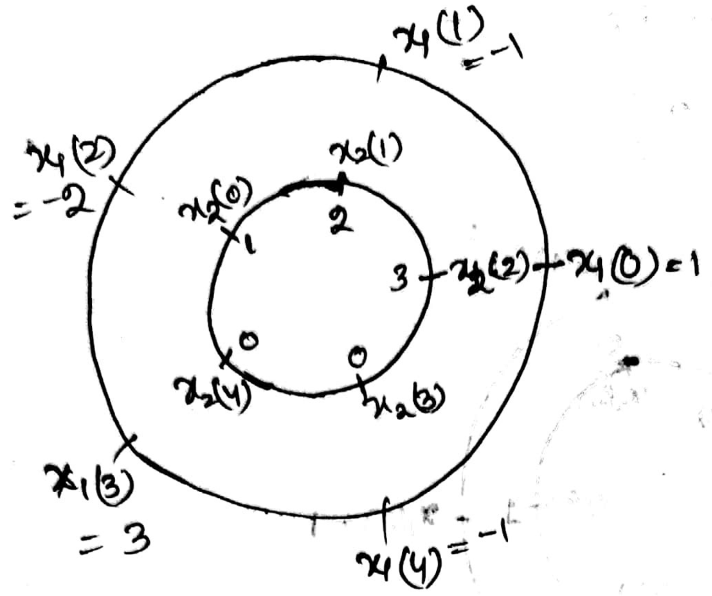
$$y(0) = 1\times 1 + 0\times(-1) + 0\times(-2) + 3\times 3 + 2\times(-1)$$
$$= 1 + 9 + (-2) = 8$$

Step-2



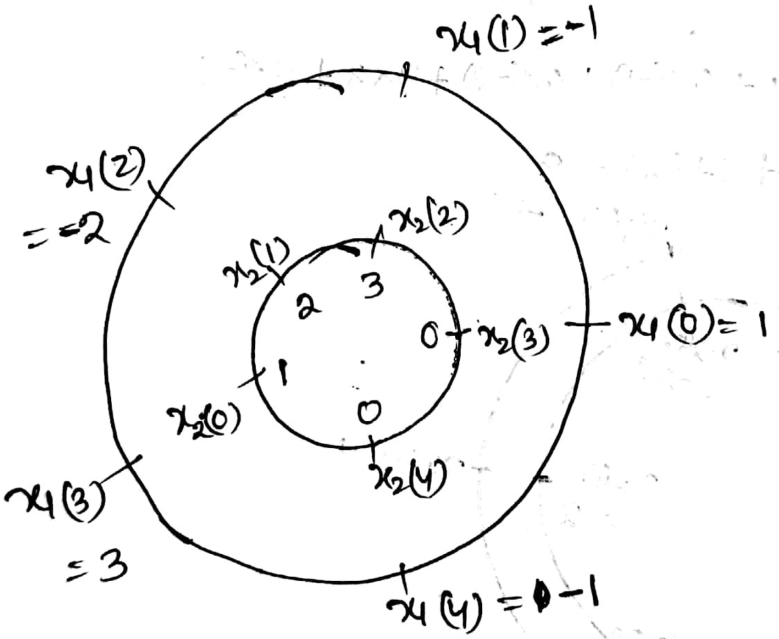
$$y(1) = 2\times 1 + 1\times(-1) + 0\times(-2) + 0\times 3 + 3\times(-1)$$
$$= \cancel{2+3=-2}$$
$$= 2 - 1 + 0 + 0 - 3 = -2$$

Step-3



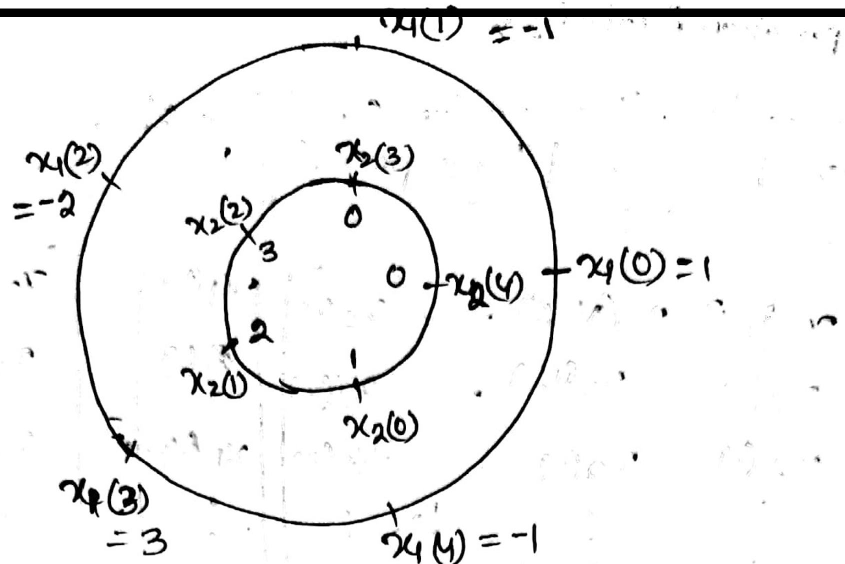
$$y(2) = 3\times 1 + 2\times(-1) + 1\times(-2) + 0\times 3 + 0\times(-1)$$
$$= 3-2-2+0+0$$
$$= -1$$

Step-4



$$y(3) = 0\times 1 + 3\times(-1) + 2\times(-2) + 1\times 3 + 0\times(-1)$$
$$= \cancel{0+(-3)-4}$$
$$= 0-3-4+3+0$$
$$= -4$$

Step-5

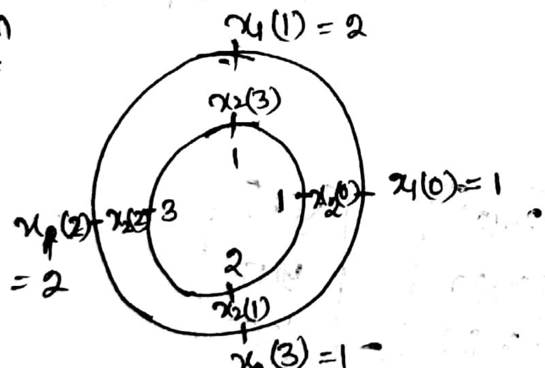


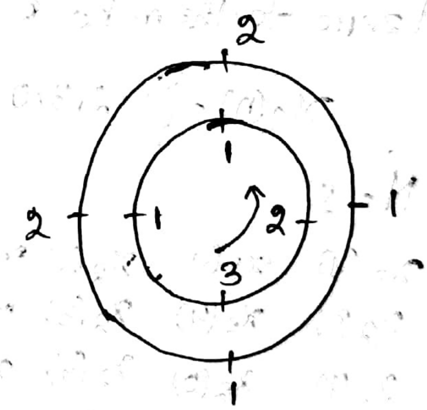
$$y(4) = 0\times1 + 0\times(-1) + 3\times(-2) + 2\times3 + 1\times(-1)$$
$$= 0+0-6+6-1$$
$$= -1$$

$$y(n) = \{8, -2, -1, -4, -1\}$$

Q21 Find the Circular Convolution of two sequences $x_1(n) = \{1,2,2,1\}$ and $x_2(n) = \{1,2,3,1\}$

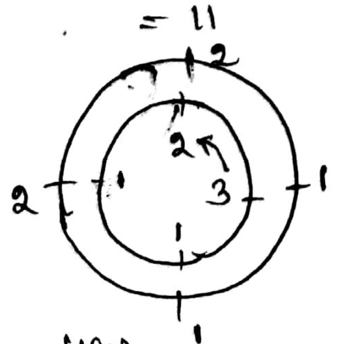
Soln

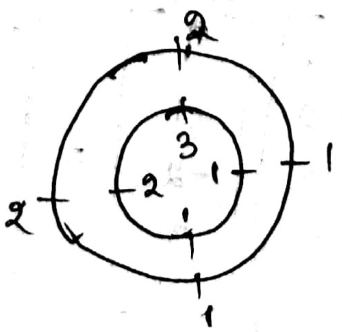




$$y(0) = 1\times1 + 1\times2 + 3\times2 + 2\times1$$
$$= 1+2+6+2$$
$$= 11$$

$$y(1) = 1\times2 + 2\times1 + 2\times1 + 3\times1$$
$$= 2+2+2+3 = 9$$





$$y(n) = \{11, 9, 9, 12\}$$ Ans

$$y(2) = 1\times3 + 2\times2 + 2\times1 + 1\times1$$
$$= 3+4+2+1 = 9$$

$$y(3) = 1\times1 + 2\times3 + 2\times2 + 1\times1$$
$$= 1+6+4+1 = 12$$

(iv) Matrix Multiplication Method :→

In this method, the circular convolution of two sequences $x_1(n)$ & $x_2(n)$ can be obtained by representing the sequences in matrix form as shown below.

$$\begin{bmatrix} x_2(0) & x_2(N-1) & x_2(N-2) & \cdots & x_2(1) \\ x_2(1) & x_2(0) & x_2(N-1) & \cdots & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & \cdots & x_2(3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_2(N-1) & x_2(N-2) & x_2(N-3) & \cdots & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ \vdots \\ x_1(N-1) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ \vdots \\ x_3(N-1) \end{bmatrix}$$

Q. Find the circular convolution ~~using~~ ~~matrix~~ of the finite duration sequences $x_1(n) = \{1, -1, -2, 3, -1\}$, $x_2(n) = \{1, 2, 3\}$ using matrix method.

Soln

Given $x_1(n) = \{1, -1, -2, 3, -1\}$

$x_2(n) = \{1, 2, 3\}$

Add zero to ~~the~~ make ~~leg~~ length of the sequences equal.

$x_2(n) = \{1, 2, 3, 0, 0\}$

$N = 5$

$$\begin{bmatrix} x_2(0) & x_2(4) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(4) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(4) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) & x_2(4) \\ x_2(4) & x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

$$\begin{array}{ccccc} x_2(n) & & x_1(n) & & y(n) \\ \begin{bmatrix} 1 & 0 & 0 & 3 & 2 & 1 \\ 2 & 1 & 0 & 0 & 3 & 2 \\ 3 & 2 & 1 & 0 & 0 & 3 \\ 0 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 0 \end{bmatrix} & & \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} & = & \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix} \end{array}$$

Zero Padding

The method of adding zeros to the sequence is known as Zero Padding.