INTEGRATION

After studying differentiation it is natural to study its inverse process. This process is called integration.

→ Integration is study for finding or calculating the area of the region bounded by the graph of the function.

Introduction

If $f(x)$ is a function

then $\frac{d}{dx} f(x) = f'(x)$

but integration when $\int f'(x)\,dx = f(x) + k$

So Integration is also known as Antiderivatives.

Ex $\frac{d}{dx} \sin x = \cos x$.

Here $\sin x$ is differentiated w.r.t x.

Here $\cos x$ is the derivative of $\sin x$.

$\int \cos x \, dx = \sin x$

→ The symbol $\int$ is used to denote the operation of integration and called the integral sign.

Here $\cos x$ is called → Integrand

dx denotes integrated w.r.t x.

→ If $f'(x)$ is the derivative of $f(x)$. (and so $f(x)+k$)

→ Then $f(x)+k$ denotes the family of all anti-derivatives of $f'(x)$.

→ Here k is indefinite constant. (Any value depending on)

Therefore $f(x)+k$ is called the indefinite integral of $f'(x)$.

We write $\int f'(x)\,dx = f(x)+k$. K being constant of integration which is arbitrary.

→ One may use any other symbol like, A, B, C ... etc. for constant of integration.

Some properties of integration

(i) $\int (f(x) \pm g(x) \pm ...)\,dx = \int f(x)\,dx \pm \int g(x)\,dx \pm \int ...$

(ii) $\int k\,f(x)\,dx = k\int f(x)\,dx$

(iii) $\int (k_1 f_1(x) + k_2 f_2(x) + k_3 f_3(x) ....)\,dx$

$= k_1\int f_1(x)\,dx + k_2\int f_2(x)\,dx + k_3\int f_3(x)\,dx ....$

Methods of integration

There are five methods of integration.

1. Integration by using standard function / by using standard formula.
2. Integration by substitution methods.
3. Integration by decomposition into sum or. By using some trigonometric function.

4. Integration by parts
5. Definite integration.

Some standard formula of integration

Derivatives

(1) $\frac{d}{dx} x = 1$

(2) $\frac{d}{dx} x^n = n x^{n-1}$

(3) $\frac{d}{dx} \sin x = \cos x$

(4) $\frac{d}{dx} \cos x = -\sin x$

(5) $\frac{d}{dx} \tan x = \sec^2 x$

(6) $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

(7) $\frac{d}{dx} \sec x = \sec x . \tan x$

(8) $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x . \cot x$

(9) $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

(10) $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$

(11) $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

(12) $\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$

(13) $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$

(14) $\frac{d}{dx} \operatorname{cosec}^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$

Integration

(1) $\int dx = x + C$

(2) $\int x^n dx = \frac{x^n}{n+1} + C$

(3) $\int \cos x \, dx = \sin x + C$

(4) $\int \sin x \, dx = -\cos x + C$

(5) $\int \sec^2 x \, dx = \tan x + C$

(6) $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$

(7) $\int \sec x . \tan x \, dx = \sec x + C$

(8) $\int \operatorname{cosec} x . \cot x \, dx = -\operatorname{cosec} x + C$

(9) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

(10) $\int \frac{1}{\sqrt{1-x^2}} dx = -\cos^{-1} x + C$

(11) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

(12) $-\int \frac{1}{1+x^2} dx = \cot^{-1} x + C$

(13) $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

(14) $-\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{cosec}^{-1} x + C$

(15) $\frac{d}{dx} e^x = e^x$

(16) $\frac{d}{dx} \log x = \frac{1}{x}$

(17) $\frac{d}{dx} a^x = a^x \log a$

(15) $\int e^x dx = e^x + c$

(16) $\int \frac{1}{x} dx = \log x + k$

(17) $\int a^x dx = \frac{a^x}{\log a} + c$

(18) $\int \tan x \, dx = \ln|\sec x| + c$

(19) $\int \cot x \, dx = \ln|\sin x| + c$

(20) $\int \sec x \, dx = \ln|\sec x + \tan x| + c$

(21) $\int \operatorname{cosec} x \, dx = \ln|\operatorname{cosec} x - \cot x| + c$

1. Integration by using standard formula / function →

→ In this section, we have to directly use the formula or convert into the formula basis.

Example -1

(1) $\int (x^6 + x^2 + x + 1) dx$

$= \int x^6 dx + \int x^2 dx + \int x \, dx + \int 1 . dx$ (Here we use $\int x^n dx = \frac{x^{n+1}}{n+1} + k$ or $\int dx = x + k$)

$= \frac{x^7}{7} + \frac{x^3}{3} + \frac{x^2}{2} + x + c$

Example - $\int \left(4 \cos x - 3e^x + \frac{2}{\sqrt{1-x^2}}\right) dx$.

use the properties of integration & take constant as outside.

(3)

$= 4\int \cos x\,dx - 3\int e^x dx + 2\int \frac{1}{\sqrt{1-x^2}} dx$

$= 4\sin x - 3e^x + 2\sin^{-1}x + K$ Ans

Example $\int 6x^3(x+5)^2 dx$

$= \int 6x^3(x^2+10x+25)\,dx = \int (6x^5 + 60x^4 + 150x^3)\,dx$

$= 6\int x^5 dx + 60\int x^4 dx + 150\int x^3 dx.$

$= \cancel{6}\frac{x^6}{\cancel{6}} + \overset{12}{\cancel{60}}\cdot\frac{x^5}{\cancel{5}} + \overset{75}{\cancel{150}}\cdot\frac{x^4}{\cancel{4}_2} + K$

$= x^6 + 12x^5 + 75x^4 + K$ Ans

Some important questions

(i) $\int \frac{\sin x}{\cos^2 x} dx$

$= \int \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} dx = \int \tan x\cdot\sec x\,dx = \sec x + C$

Ans

(ii) $\int \frac{1}{1-\cos^2 x} dx = \int \frac{1}{\sin^2 x + \cancel{\cos^2 x} - \cancel{\cos^2 x}} dx$

$= \int \frac{1}{\sin^2 x} dx = \int \operatorname{cosec}^2 x\,dx = -\cot x + C.$

(iii) $\int \frac{\sin^2 x}{1+\cos x} dx = \int \frac{1-\cos^2 x}{1+\cos x} dx = \int \frac{(1-\cos x)\cancel{(1+\cos x)}}{\cancel{(1+\cos x)}} dx$

$= \int (1-\cos x)\,dx = \int 1\,dx - \int \cos x\,dx$

$= x - \sin x + K$ Ans

(2) → Integration by substitution method

When the integrand is not in a standard form, it can sometimes be transformed to integrable form by a suitable substitution.

There is no fixed formula for substitution.

→ Any symbol for variable like, s, t, u, v, w, x, y, z may be chosen for substitution other than the variable of the given integral.

→ However, after integration is over, the original variable should be put back.

Example

$$\int (ax+b)^n \, dx$$

There is no direct formula for it so we take

$ax+b = z$

then differentiate / derivative both sides w.r.t x.

$$\frac{d}{dx}(ax+b) = \frac{d}{dx} z$$

$$\rightarrow a = \frac{dz}{dx} \Rightarrow dx = \frac{dz}{a}$$

Hence. $\int z^n \frac{dz}{a}$

$$= \frac{1}{a} \int z^n dz = \frac{1}{a} \frac{z^{n+1}}{n+1}$$

Now putting back for z

$$= \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + k \text{ Ans}$$

dz

(4) **Example** $\int \cos(ax+b)\,dx$ (we know $\int \cos x\,dx = \sin x + c$ but we donot know $\int \cos(ax+b)\,dx$)

So we take substitute $ax+b = z$

$$\frac{d}{dx}(ax+b) = \frac{dz}{dx}$$

$$a = \frac{dz}{dx} \Rightarrow dx = \frac{dz}{a}$$

$$= \int \cos z \frac{dz}{a} = \frac{1}{a}\int \cos z\,dz = \frac{1}{a}\sin z + K$$

$$= \frac{1}{a}\sin(ax+b) + K$$

Ex

$\int \frac{g'(x)}{g(x)}\,dx$. Always put $g(x) = z$

then we get $\int \frac{dz}{z} = \ln|z| + k$

- Such as

Ex $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx.$

let $\sin x = v$

$\cos x = \frac{dv}{dx} \Rightarrow \cos x\,dx = dv$

$$= \int \frac{dv}{v} = \ln|v| + K = \ln|\sin x| + k.$$

Ex $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

Let $\cos x = z \Rightarrow -\sin x = \frac{dz}{dx} \Rightarrow -\sin x\,dx = dz$

$$= \int -\frac{dz}{z} = -\ln|z| + k = -\ln|\cos x| + k$$

Ans.

Ex $\int \sin^7 x . \cos x \, dx$ (power value take as the substitute)

Here $\sin x = z$

$\cos x = \frac{dz}{dx} \Rightarrow \cos x \, dx = dz$

$= \int z^7 dz = \frac{z^8}{8} + k$

$= \frac{\sin^8 x}{8} + k$ Ans

Ex:- $\int 2 \, e^{\tan^2 x} \tan x . \sec^2 x \, dx.$

Putting $\tan^2 x = z$

$2 \tan x . \sec^2 x = \frac{dz}{dx} \Rightarrow 2 \tan x . \sec^2 x \, dx = dz$

$= \int e^z dz = \int e^z dz$

$= e^z + k = e^{\tan^2 x} + k.$

Ex:- $\int \frac{(\tan^{-1} x)^3}{1+x^2} dx$ In this case biggest power is 3

so $\tan^{-1} x = z$

$\frac{1}{1+x^2} = \frac{dz}{dx}$

$\frac{1}{1+x^2} dx = dz$

$\Rightarrow \int z^3 dz = \frac{z^4}{4} + k = \frac{(\tan^{-1} x)^4}{4} + k$

Ex $\int e^{2x+7} dx$, Putting $2x+7 = z$

$2 = \frac{dz}{dx} \Rightarrow dx = \frac{dz}{2}$

$= \int e^z \frac{dz}{2} = \frac{1}{2} \int e^z dz = \frac{1}{2} e^z + k = \frac{1}{2} e^{2x+7} + k.$

(3) Integration by decomposition into sum / Integration of some trigonometric function

(1) To reduce it to the sum of sines or cosines of multiples angles.

① $\sin mx \cos nx = \frac{1}{2} \cdot 2 \sin mx \cdot \cos nx$

$= \frac{1}{2}\left[\sin(m+n)x + \sin(m-n)x\right]$

(2) $\sin mx \cdot \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$

(3) $\cos mx \cdot \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$

Example :- $\int \sin 3x \cdot \cos 2x \, dx$

$= \frac{1}{2}\int(\sin 5x + \sin x)dx$

$= \frac{1}{2}\left[\int \sin 5x \, dx + \int \sin x \, dx\right]$

$= \frac{1}{2}\left[-\frac{1}{5}\cos 5x - \cos x\right] + k$

$= -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + k$

Example $\int \cos 4x \cdot \cos 3x \, dx$

$= \frac{1}{2}\left[\int(\cos 7x + \cos x)\,dx\right]$

$= \frac{1}{2}\left[\int \cos 7x \, dx + \frac{1}{2}\int \cos x \, dx\right]$

$= \frac{1}{2}\frac{1}{7}\sin 7x + \frac{1}{2}\sin x + k$

$= \frac{1}{14}\sin 7x + \frac{1}{2}\sin x + k$

(9) Integration by parts

Ex→ $\int \sin^3 x\, dx = \int \sin^2 x \cdot \sin x\, dx$

$= \int (1-\cos^2 x) \sin x\, dx$

Let $\cos x = z$

$-\sin x = \frac{dz}{dx}$

$\sin x\, dx = -dz$

$= \int (1-z^2) \cdot -dz$

$= -\int dz + \int z^2 dz$

$= -z + \frac{z^3}{3} + k = -\cos x + \frac{\cos^3 x}{3} + k$

Ex:- $\int \sin^4 x \cos^3 x\, dx$

$= \int \sin^4 x \cdot \cos^2 x \cdot \cos x\, dx$

$= \int \sin^4 x (1-\sin^2 x) \cdot \cos x\, dx$.

Let $\sin x = z \rightarrow \cos x = \frac{dz}{dx} \Rightarrow \cos x\, dx = dz$

$= \int z^4 (1-z^2)\, dz$

$= \int z^4 dz - \int z^6 dz$

$= \frac{z^5}{5} - \frac{z^7}{7} + k$

$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + k$

Ex:- $\int \frac{\cos^3 x}{\sin^4 x} dx = \int \frac{\cos^2 x}{\sin^4 x} \cdot \cos x\, dx$

$= \int \frac{1-\sin^2 x}{\sin^4 x} \cdot \cos x\, dx$.

Let $\sin x = z$

$\cos x = \frac{dz}{dx} \Rightarrow \cos x\, dx = dz$

$= \int \frac{1-z^2}{z^4} dz \Rightarrow \int z^{-4} dz - \int z^{-2} dz = \frac{z^{-3}}{-3} - \frac{z^{-1}}{-1} + k$

$= -\frac{1}{3z^3} + \frac{1}{z} + k = -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + k$.

(6) (5) Integration by parts

When there is the product of two function is given then we have to use integration by parts.

In derivatives

$$\frac{d}{dx}(uv) = u\frac{d}{dx}v + v.\frac{d}{dx}u.$$

But in integration

$$\int(u.v)\,dx$$

$$= u.\int v\,dx - \int\left(\frac{d}{dx}u.\int v\,dx\right)dx$$

This rule is called integration by parts.

Or. first function $\times \int$ second function

$- \int\left(\frac{d}{dx}\text{ first function}.\int\text{second function}\right)dx.$

~~Such~~ Just like as

$$\int x.\,e^x\,dx.$$

Here x is a function & e^x is also function. so we have to use integration by parts.

$$\int x.\,e^x\,dx = x\int e^x\,dx - \int\left(\frac{d}{dx}x.\int e^x\,dx\right)dx.$$

$$= x.\,e^x - \int e^x\,dx \quad \left(\because \frac{d}{dx}x = 1 \;\&\; \int e^x dx = e^x\right)$$

$$= x.\,e^x - e^x + k$$ Ans

But if we take e^x as the first function.

(7)

then there is some difficult comes out.

$$\int x \cdot e^x \, dx = e^x \int x \, dx - \int \left(\frac{d}{dx} e^x \int x \, dx \right) dx$$

$$= e^x \cdot \frac{x^2}{2} - \int e^x \frac{x^2}{2} \, dx$$

∴ which create more difficult so we have to chose the function. clearly & suitable manner.

Usually we use ILATE Rule to ~~the~~ take the first function.

ILATE means.

I → Inverse function.

L → Logarithm function.

A → Algebraic function.

T → Trigonometric function

E → Exponential function.

But simply the value which can be easily derivable & decreases. That value should be taken as first function.

Function to be integrated	First function	Second function
$x^n e^x$	x^n	e^x
$x^n \cos x$	x^n	$\cos x$
$x^n \sin x$	x^n	$\sin x$.
$x^n (\ln x)^m$	$(\ln x)^m$	x^n
$x^n \sin^{-1} x$	$\sin^{-1} x$	x^n
$x^n \tan^{-1} x$	$\tan^{-1} x$	x^n.

Lireye.

(7) Example (i) $\int x \cos x \, dx$ (ILATE Rule)

$= x \int \cos x \, dx - \int \left(\frac{d}{dx} x \int \cos x \, dx \right) dx$

$= x \cdot \sin x - \int \sin x \, dx$

$= x \sin x - \int \sin x \, dx$

$= x \sin x + \cos x + K$

Example - 2

$\int x^2 e^x \, dx$ (ILATE Rule)

$= x^2 \int e^x \, dx - \int \left(\frac{d}{dx} x^2 \int e^x \, dx \right) dx$

$= x^2 \cdot e^x - \int 2x \cdot e^x \, dx$

$= x^2 \cdot e^x - 2 \int x \cdot e^x \, dx$

$= x^2 \cdot e^x - 2 \left[x \cdot \int e^x \, dx - \int \left(\frac{d}{dx} x \cdot \int e^x \, dx \right) dx \right]$

$= x^2 \cdot e^x - 2 \left[x \cdot e^x - \int e^x \, dx \right]$

$= x^2 \cdot e^x - 2x \cdot e^x + 2e^x + K$

$= e^x (x^2 - 2x + 2) + K.$

Example $\int \tan^{-1} x \, dx$

$= \int 1 \cdot \tan^{-1} x \, dx = \tan^{-1} x \int 1 \, dx - \int \left(\frac{d}{dx} \tan^{-1} x \cdot \int 1 \, dx \right) dx$

$= \tan^{-1} x \cdot x - \int \frac{1}{1+x^2} \cdot x \, dx$

$= x \cdot \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$

$= x \cdot \tan^{-1} x - \frac{1}{2} \ln |1+x^2| + K.$

When $\sin^{-1} x$, $\cot^{-1} x$, $\tan^{-1} x$, etc or $\log x$ is present alone in the integrand, $1 = x^0$ has to be taken as the second function

I can get the whole ... of circle.

functions by putting $x = a\sin\theta$ ...

Ex:- $\int e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$

$$= \int \frac{e^x}{1+\cos x}\, dx + \int e^x \frac{\sin x}{1+\cos x}\, dx$$

$$= \int \frac{e^x}{1+\cos x}\, dx + \left(\frac{\sin x}{1+\cos x} \cdot \int e^x\, dx - \int \left(\frac{d}{dx} \frac{\sin x}{1+\cos x} \int e^x\, dx\right) dx \right)$$

$$= \cancel{\int \frac{e^x}{1+\cos x}\, dx} + e^x \frac{\sin x}{1+\cos x} - \cancel{\int \frac{1}{1+\cos x} \cdot e^x\, dx}.$$

$$= e^x \frac{\sin x}{1+\cos x} + k.$$

Ex $\int e^x \sin x\, dx$.

let $I = \int e^x \sin x\, dx$

$$= e^x \int \sin x\, dx - \int \left(\frac{d}{dx} e^x \int \sin x\, dx\right) dx$$

$$= -e^x \cos x + \int e^x \cos x\, dx$$

$$= -e^x \cos x + \left(e^x \int \cos x\, dx - \int \left(\frac{d}{dx} e^x \int \cos x\, dx\right) dx\right)$$

$$= -e^x \cos x + \left(e^x \sin x - \int e^x \sin x\, dx\right)$$

$$= -e^x \cos x + e^x \sin x - I$$

$$2I = -e^x \cos x + e^x \sin x$$

$$I = \frac{-e^x \cos x + e^x \sin x}{2}$$

$$\int e^x \sin x\, dx = \frac{1}{2}\left(-e^x \cos x + e^x \sin x\right)$$

Ans

(8)

Integration by trigonometric substitution

→ The irrational forms $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$, $\sqrt{x^2-a^2}$ can be simplified or radical free functions by putting $x = a\sin\theta$, $x = a\tan\theta$, $x = a\sec\theta$ respectively (or $x = a\cos\theta$, $x = a\cot\theta$, $x = a\,\mathrm{cosec}\,\theta$).

→ The substitution $x = a\tan\theta$ (or $x = a\cot\theta$) can be useful in case of presence of x^2+a^2 in the integrand, particularly when it is present in the denominator.

Example

(i) $\int \frac{dx}{\sqrt{a^2-x^2}}$ we know $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$

Let $x = a\sin\theta$, so that $\theta = \sin^{-1} x/a$

$dx = a\cos\theta\, d\theta$

$$\int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$$

$$= \int \frac{a\cos\theta}{\sqrt{a^2(1-\sin^2\theta)}}\, d\theta = \int \frac{\cancel{a}\cancel{\cos\theta}}{\cancel{a}\cancel{\cos\theta}}\, d\theta$$

$$= \int d\theta = \theta + k$$

$$= \sin^{-1} x/a + k$$

(ii) $\int \frac{dx}{x^2+a^2}$ Let $x = a\tan\theta$ because $\int \frac{dx}{1+x^2} = \tan^{-1}x + c$

$dx = a\sec^2\theta\, d\theta$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2\tan^2\theta + a^2}$$

$$= \int \frac{a\sec^2\theta \, d\theta}{a^2(\tan^2\theta + 1)}$$

$$= \int \frac{\cancel{a}\,\cancel{\sec^2\theta}}{a^{\cancel{2}}\,\cancel{\sec^2\theta}} \, d\theta$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + k$$

$$= \frac{1}{a}\tan^{-1}x/a + k$$

(iii) Let $x = a\tan\theta$, so that $dx = a\sec^2\theta \, d\theta$

$$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{\cancel{a}\,\sec^{\cancel{2}}\theta}{\cancel{a}\,\cancel{\sec\theta}} \, d\theta$$

$$= \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + c$$

(iv) Let $x = a\sec\theta$, so that

$dx = a\sec\theta.\tan\theta \, d\theta$

$$\therefore \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta.\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}} = \int \frac{\cancel{a}\,\sec\theta.\cancel{\tan\theta}}{\cancel{a}\,\cancel{\tan\theta}} \, d\theta$$

$$= \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + k$$

$$= \ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + c$$

Example

(1) $\displaystyle\int \frac{dx}{\sqrt{25 - 16x^2}}$

$$= \frac{1}{4}\int \frac{dx}{\sqrt{\left(\frac{5}{4}\right)^2 - x^2}} = \frac{1}{4}\sin^{-1}\frac{x}{5/4} + c$$

$$= \frac{1}{4}\sin^{-1}\frac{4x}{5} + c \quad \text{Ans.}$$

(7)

(ii) Ex- $\int \frac{dx}{x\sqrt{x^8-4}} = \frac{1}{4}\int \frac{4x^3}{x^4\sqrt{x^8-4}}\,dx$

$= \frac{1}{4}\int \frac{dz}{z\sqrt{z^2-2^2}}$ putting $x^4 = z$, $4x^3dx = dz$

$= \frac{1}{4}\cdot\frac{1}{2}\sec^{-1}\frac{z}{2} + c$

$= \frac{1}{8}\sec^{-1}\left(\frac{x^4}{2}\right) + c.$

Ex $\int \sqrt{a^2-x^2}\,dx = \int 1.\sqrt{a^2-x^2}\,dx$

$= \sqrt{a^2-x^2}\cdot\int 1dx - \int\left(\frac{d}{dx}\sqrt{a^2-x^2}\cdot\int 1\,dx\right)dx.$

$= \sqrt{a^2-x^2}\cdot x + \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx$

$= \sqrt{a^2-x^2}\cdot x + \int \frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}\,dx$

$= x\sqrt{a^2-x^2} + a^2\int\frac{dx}{\sqrt{a^2-x^2}} - \int\sqrt{a^2-x^2}\,dx$

$\therefore 2\int\sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + a^2\int\frac{dx}{\sqrt{a^2-x^2}}$

$= x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a} + C$

$= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$ Ans.

Ex $\int\sqrt{x^2+a^2}\,dx.$

(ii) $\int\sqrt{x^2-a^2}\,dx$ can be done similarly.

Definite Integral

→ In the previous sections, we have studied about the indefinite integrals and discussed few methods of them including integral of some special function.

→ In this section, we study definite integral.

→ A definite integral is denoted by $\int_a^b f(x)\,dx$.

→ Here a is called lower limit of the integral and b is called the upper limit of the integral.

Properties of Definite Integral

(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(ii) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(z)\,dz$

(iii) $\int_a^b f(x)\,dx = \int_a^{\alpha} f(x)\,dx + \int_{\alpha}^b f(x)\,dx$, $a < \alpha < b$

Example :-

(i) $\int_1^4 [x]\,dx = \int_1^2 [x]\,dx + \int_2^3 [x]\,dx + \int_3^4 [x]\,dx$

$= \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^4 3\,dx$

$= 2-1+2(3-2)+3(4-3) = 6$ Ans

(ii) $\int_{-3}^4 |x|\,dx = \int_{-3}^0 |x|\,dx + \int_0^4 |x|\,dx$

$= \int_{-3}^0 (-x)\,dx + \int_0^4 x\,dx = \int_0^3 x\,dx + \int_0^4 x\,dx$

$= \left|\frac{x^2}{2}\right|^{-3} + \left|\frac{x^2}{2}\right|^4 = \frac{9}{2} - 0 + \frac{16}{2} - 0 = \frac{25}{2}$

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More Properties of Definite Integral

(i) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(ii) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f \text{ is even function} \\ 0 & \text{if } f \text{ is odd function.} \end{cases}$

(iii) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

Ex- $\int_0^{\pi/2} \frac{dx}{1+\tan x}$

Let $I = \int_0^{\pi/2} \frac{dx}{1+\tan(\pi/2 - x)}$ $\left(\text{As } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$= \int_0^{\pi/2} \frac{dx}{1+\cot x}$.

Now $I + I = \int_0^{\pi/2} \frac{dx}{1+\tan x} + \int_0^{\pi/2} \frac{dx}{1+\cot x}$

$= \int_0^{\pi/2} \frac{(1+\cot x)(1+\tan x)}{(1+\tan x)(1+\cot x)}\,dx$

$= \int_0^{\pi/2} dx$

$2I = [x]_0^{\pi/2}$

$2I = \pi/2$

$I = \pi/4$

$\therefore \int_0^{\pi/2} \frac{dx}{1+\tan x} = \pi/4$

$\underline{Ex}$: $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$

$\therefore$ Let $\displaystyle I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}\,dx$

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx$$

Now $\displaystyle I + I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx + \int_0^{\pi/4} \frac{\cos x}{\cos x + \sin x}\,dx$

$$= \int_0^{\pi/4} \frac{(\sin x + \cos x)}{(\sin x + \cos x)}\,dx$$

$$= \int_0^{\pi/2} dx$$

$$2I = \left[x\right]_0^{\pi/2}$$

$$2I = \pi/2$$

$$I = \pi/4$$

$\therefore$ $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \pi/4$

Important question of definite integrals

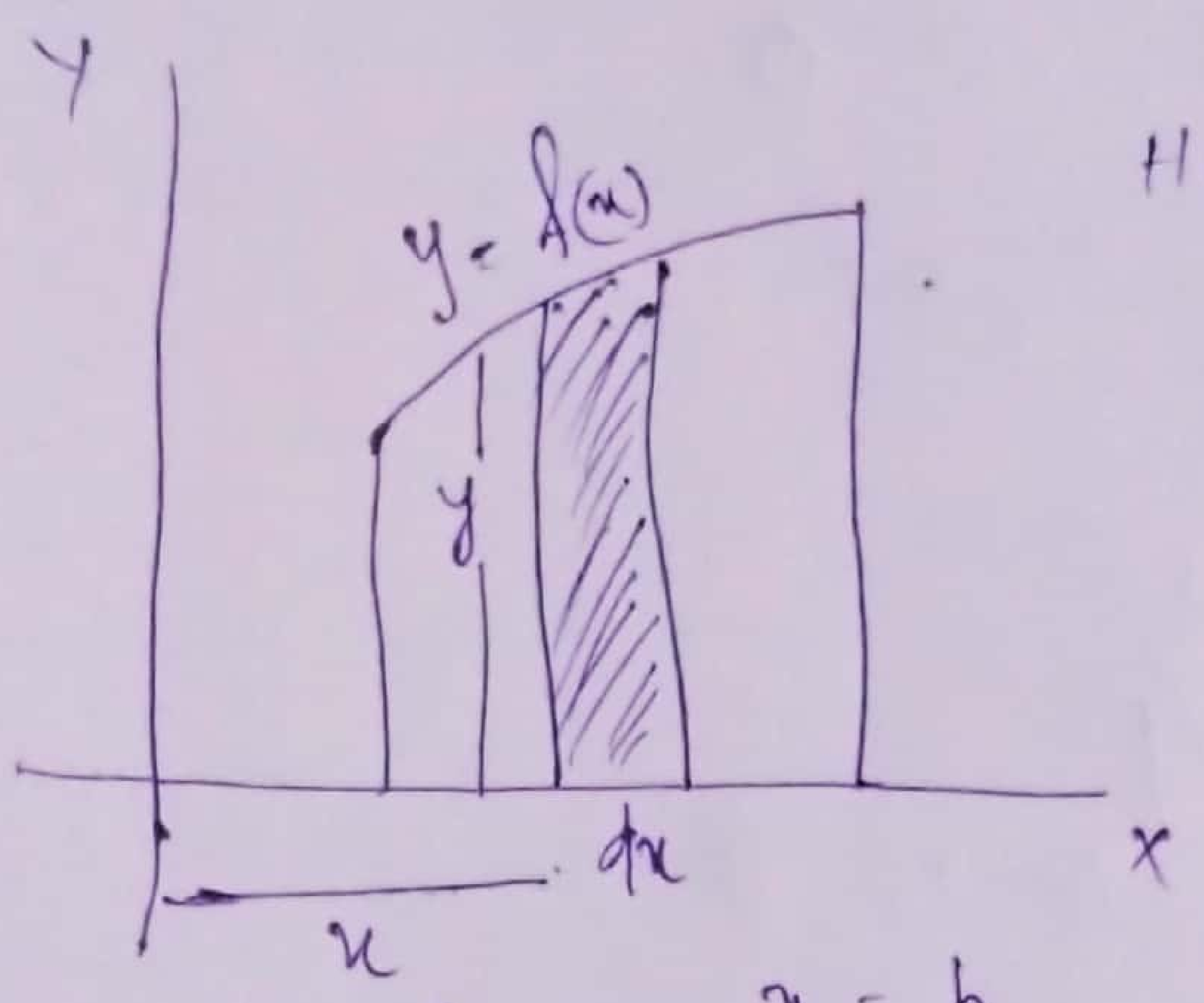
(1) $\displaystyle\int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$ (2) $\displaystyle\int_0^{\pi} \frac{x\,dx}{1+\sin x}$ (3) $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$

(4) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx$ (5) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\cot x}$. (6) $\displaystyle\int_0^{1} \frac{\ln(1+x)}{1+x^2}\,dx$ ($x = \tan\theta$)

Application of integration

→ **Area Enclosed by a curve & X-axis**

→ In this section we have to find the area of the enclosed curve.



Here $dA = y\,dx$

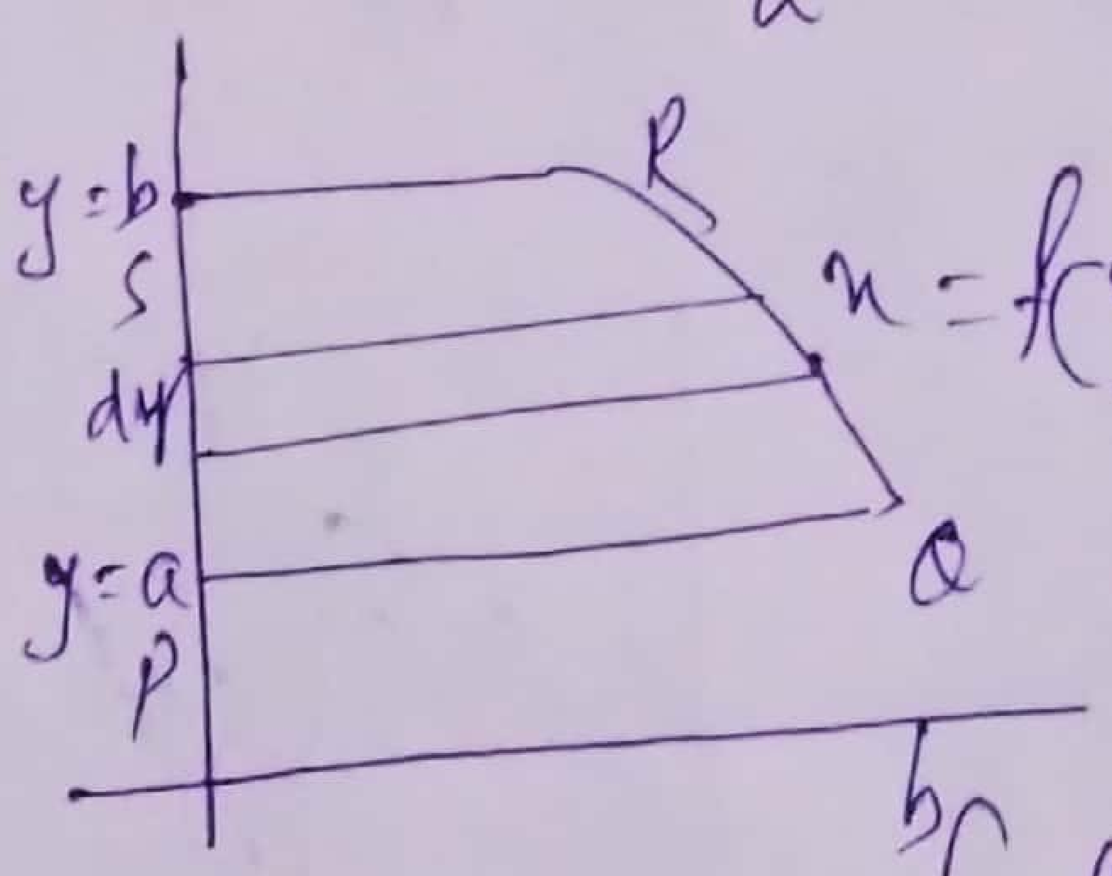
$dA = f(x)\,dx$.

$x = a$, $x = b$

By integrating $\int_a^b dA = \int_a^b f(x)\,dx$.

$$A = \int_a^b f(x)\,dx = f(b) - f(a)$$

and if

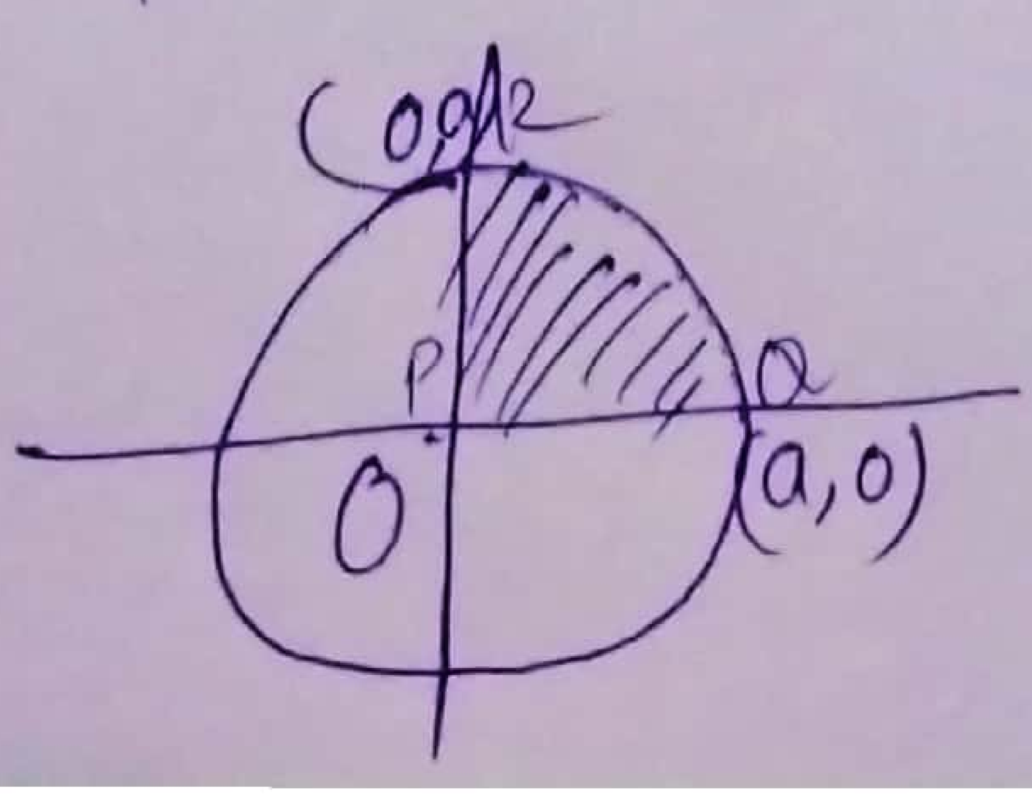


Here Area $= A = \int_a^b f(y)\,dy$

before we know $\boxed{\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C}$

Example: Find the area of the circle. $x^2 + y^2 = a^2$

Ans:



If I find ~~one part of~~ area of one area then I can get the whole area of circle.

By the formula

Area $= A = \int_a^b f(x)\,dx = \int_0^a \sqrt{a^2-x^2}\,dx$ (Here $y^2 = a^2 - x^2$, $y = \sqrt{a^2-x^2}$)

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \left[\frac{a^2}{2}\sin^{-1}1 - \frac{a^2}{2}\sin^{-1}0\right]$$

$$= \frac{a^2}{2} \times \pi/2 = \frac{\pi a^2}{4}$$

Now Area of whole circle is $\cancel{4}$.

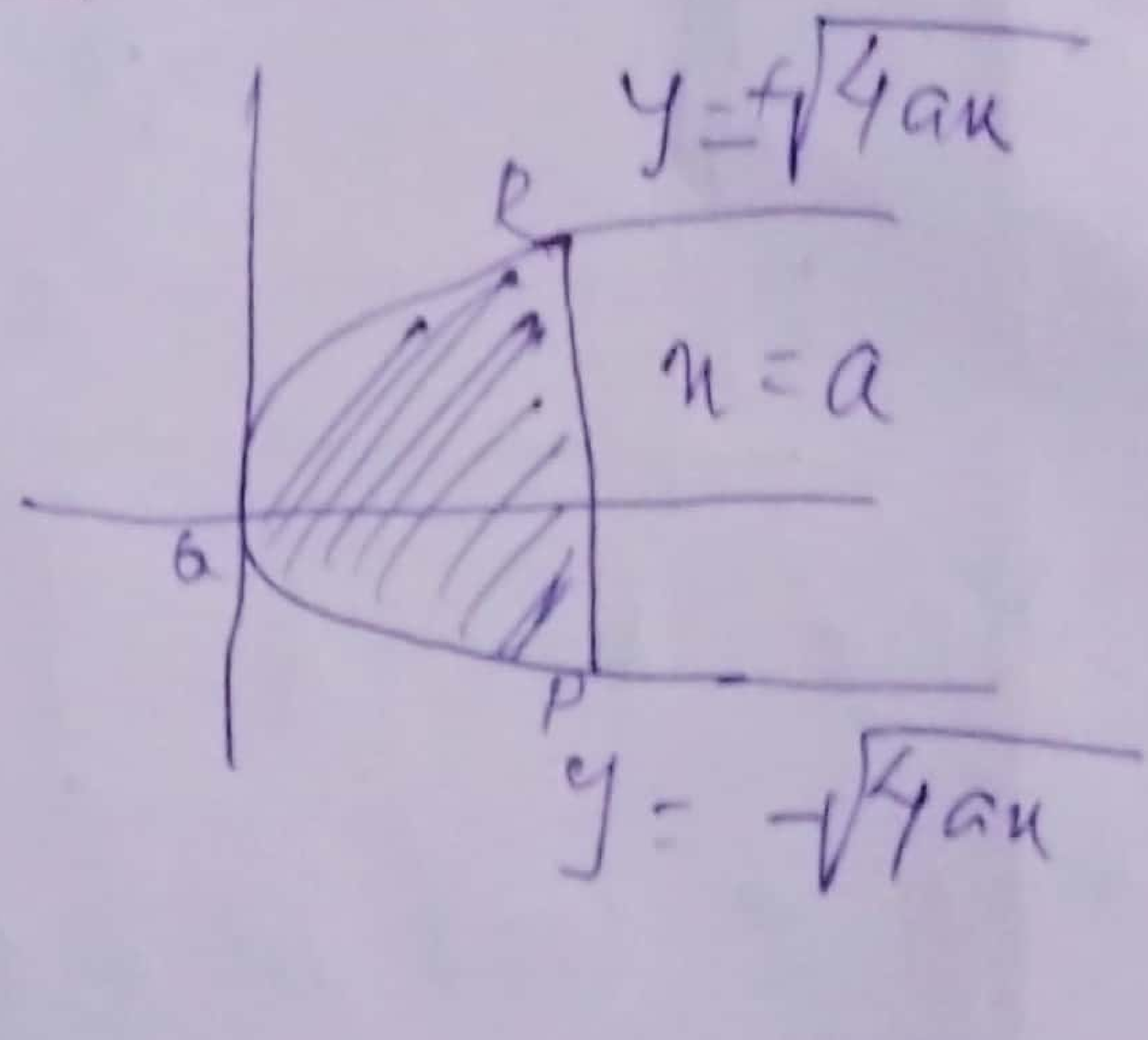
$$4 \times \text{area } PQR = \cancel{4} \times \frac{\pi a^2}{\cancel{4}}$$

$$= \pi a^2 \text{ square units}$$

Ex – To find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum $x = a$.

Ans: We observe that the curve is symmetrical about x-axis.

It lies $x = 0$, & $x = a$.



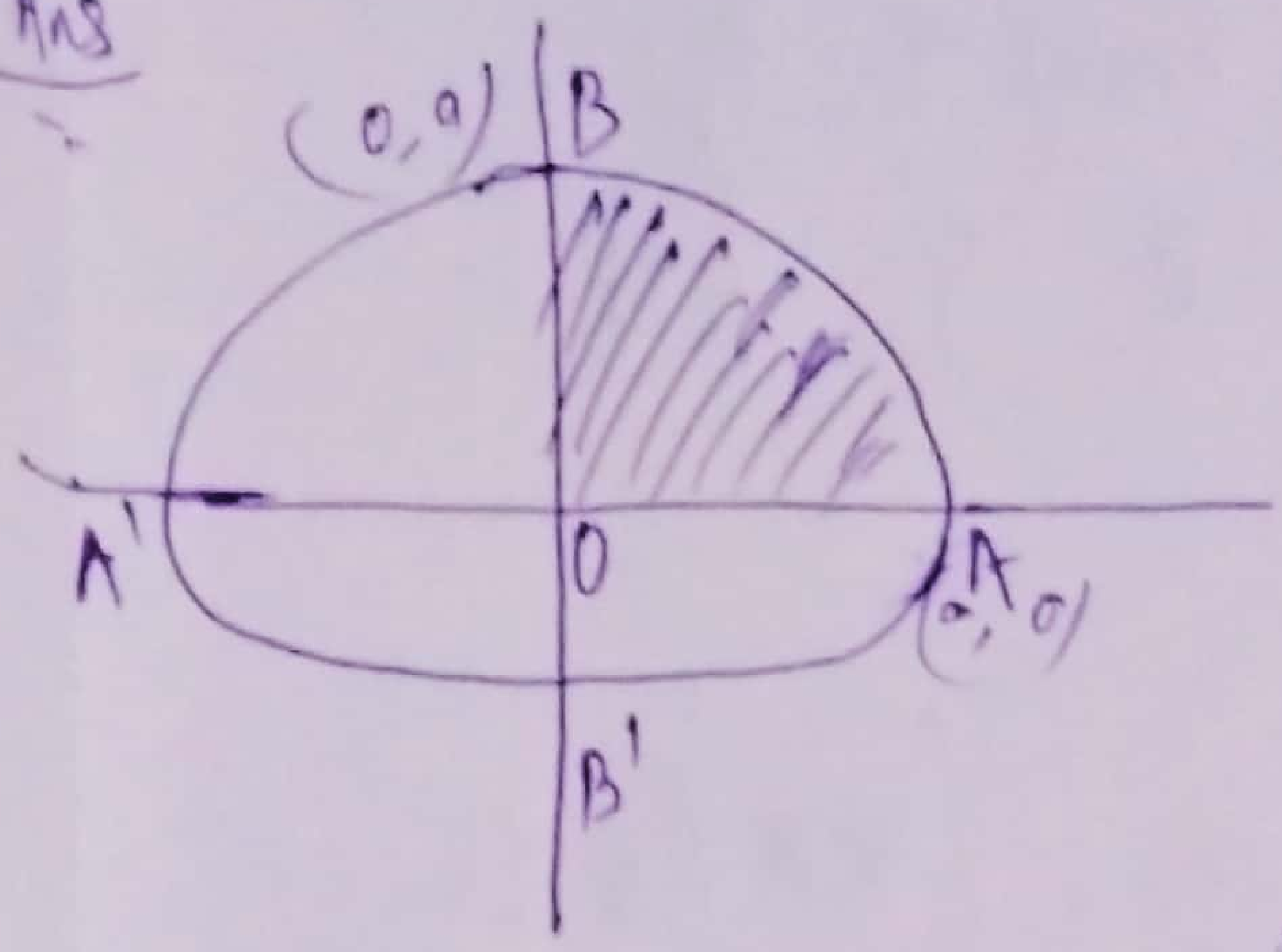
$$\text{Area of } A = 2\int_0^a f(x)\,dx$$

$$= 2\int_0^a \sqrt{4ax}\,dx$$

$$= 4\sqrt{a}\,\frac{2}{3}\left[x^{3/2}\right]_0^a = \frac{8}{3}a^2 \text{ square units.}$$

(12) Ex: To find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Ans



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow \frac{y^2}{b^2}=\frac{a^2-x^2}{a^2}$$

$$\Rightarrow y^2=\frac{b^2(a^2-x^2)}{a^2}$$

$$\Rightarrow y=\frac{b}{a}\sqrt{a^2-x^2}$$

Now Area of OAB

$$=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=\frac{b}{a}\left[\frac{a^2}{2}\sin^{-1}1\right]=\frac{b}{\cancel{a}}\,\frac{a^{\cancel{2}}}{2}\cdot\frac{\pi}{2}=\frac{\pi ab}{4}$$

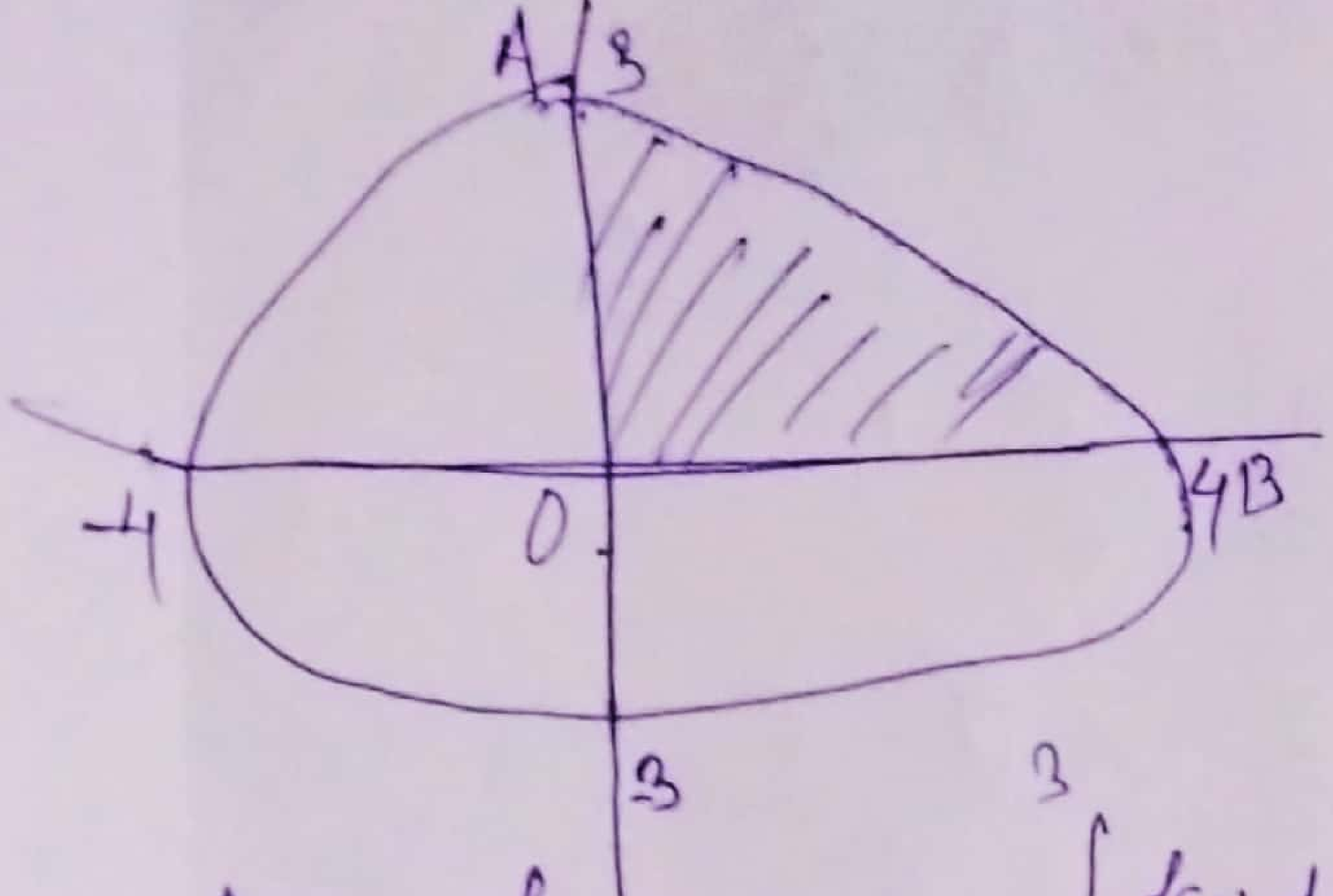
Now area of ellipse is $= 4\times$ area of OAB

$$=\cancel{4}\times\frac{\pi ab}{\cancel{4}}=\pi ab \text{ square units}$$

Ex:- Find the area of ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

Ans Here $\frac{y^2}{9}=\frac{4^2-x^2}{4^2}$ When $x=0,\ y=\pm 3$

$y=\frac{3}{4}\sqrt{4^2-x^2}$ $x=\pm 4,\ y=0$



we know $x = f(y)$

& $y = f(x)$

Area of AOB $= \int_0^3 f(y)\,dy$ or $\int_0^4 f(x)\,dx.$

$= \int_0^4 \frac{3}{4}\sqrt{4^2 - x^2}\,dx.$

$= \frac{3}{4}\int_0^4 \sqrt{4^2 - x^2}\,dx$

$= \frac{3}{4}\left[\frac{x}{2}\sqrt{4^2-x^2} + \frac{4^2}{2}\sin^{-1}\frac{x}{4}\right]_0^4$

$= \frac{3}{4}\left[\frac{16}{2}\cdot\pi/2\right] = \frac{3}{\cancel{4}} \times \frac{\cancel{16}\pi}{\cancel{4}} = 3\pi$ square units

Now we need to find the area of whole parts of the ellipse.

∴ Area of ellipse is 4 X AOB area

$= 4 \times 3\pi = 12\pi$ square units.

Some important other questions

(1) Find the area of the circle. $x^2 + y^2 = 16$

(2) Find the area of the ellipse $\frac{x^2}{25} + \frac{y^2}{36} = 1$

(3) Find the area of the enclosed parabolas $y^2 = 4ax$ and $x^2 = 4ay$.