ENGINEERING MATHEMATICS -I
FOR DIPOLMA STUDENTS

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CONTENTS

CHAPTER – 1

DETERMINANT

INTRODUCTION :

The study of determinants was started by Leibnitz in the concluding portion of seventeenth century. This was latter developed by many mathematician like Cramer, Lagrange, Laplace, Cauchy, Jocobi. Now the determinants are used to study some of aspects of matrices.

Determinant : If the linear equations

$a_1x + b_1 = 0$

and $a_2x + b_2 = 0$

have the same solution, then $\dfrac{b_1}{a_1} = \dfrac{b_2}{a_2}$

or $a_1b_2 – a_2b_1 = 0$

The expression $(a_1b_2 – a_2b_1)$ is called a **determinant** and is denoted by symbol.

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ or by (a_1b_2) where a_1, a_2, b_1 & b_2 are called the elements of the **determinant**. The elements in the horizontal direction from rows, and those in the vertical direction form **columns**. The determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ has two rows and two coloums. So it is called a **determinant of the second order** and it has 2! = 2 terms in its expansion of which one is positive and other is negative. The diagonal term, or the leading term of the determinant is a_1b_2 whose sign is positive.

Again if the linear equations

$a_1x + b_1y + c_1 = 0$ ....(i)

$a_2x + b_2y + c_2 = 0$ ....(ii)

$a_3x + b_3y + c_3 = 0$ ....(iii)

have the same solutions, we have from the last two equations by cross-multiplication.

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{1}{a_2b_3 - a_3b_2}$$

$$\text{or}\quad x = \frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2},\ y = \frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2}$$

These values of x and y must satisfy the first equation. Hence $a_1(b_2c_3 – b_3c_2) + b_1 (c_2a_3 – c_3a_2) + c_1 (a_2b_3 – a_3b_2)$

or $a_1b_2c_3 – a_1b_3c_2 + a_3 b_1c_2 – a_2b_1c_3 + a_2b_3c_1 – a_3b_2c_1$ is denoted by the symbol

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ or by $(a_1 b_2 c_3)$ and has three rows, and three columns. So it is called a **determinant of the third order** and it has 3! = 6 terms of which three terms are positive, and three terms are negative.

MINORS

Minors : The determinant obtained by suppressing the row and the column in which a particular element occurs is called the minor of that element.

Therefore, in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

the minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, that of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$ and that of c_3 is $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and so on.

The minor of any element in a third order determinant is thus a second order determinant.

The minors of $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ are denoted by $A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3$ respectively.

Hence $A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, A_2 = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \; A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

$B_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}, \; B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

$C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, C_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, \; C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

If A stands for the value of the determinant, then

$A = a_1A_1 - b_1B_1 + c_1C_1 = a_1A_1 - a_2A_2 + a_3A_3$

Cofactors : The cofactor of any element in a determinant is its coefficient in the expansion of the determinant. It is therefore equal to the corresponding minor with a proper sign.

For calculation of the proper sign to be attached to the minor of the element, one has to consider $(-1)^{i+j}$ and to multiply this sign with the minor of the element a_{ij} where i and j are respecively the row and the column to which the element a_{ij} belongs.

Thus $C_{ij} = (-1)^{i+j} M_{ij}$ Where C_{ij} and M_{ij} are respectively the cofactor and the minor of the element a_{ij}.

The cofactor of any element is generally denoted by the corresponding capital letter.

Thus for the determinant $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, cofactor of a_1 is

$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, that of b_1 is $B_1 = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}$

that of c_1 is $C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

(The sign is $(-1)^{1+3} = 1$), and so on.

We see that minors and cofactors are either equal of differ in sign only.

With this notation the determinant may be expanded in the form,

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1A_1 + b_1B_1 + c_1C_1$

Similarly we express $= a_2A_2 + b_2B_2 + c_2C_2$

$=a_3A_3 + b_3B_3 + c_3C_3$

By expanding with respect to the elements of the first column, we can write

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$= a_1A_1 + a_2A_2 + a_3A_3$

Similarly $= b_1B_1 + b_2B_2 + b_3B_3$

$= c_1C_1 + c_2C_2 + c_3C_3$

Thus the determinant can be expressed as the sum of the product of the elements of any row (or column) and the corresponding cofactors of the respective elements of the same row (or column).

PROPERTIES OF DETERMINANT

I. The value of a determinant is unchanged if the rows are written as columns and columns as rows.

If the rows and coloums are interchanged in the determinant of 2nd order $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, the determinant becoems $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

Each of the two $= a_1b_2 - a_2b_1$

$$\therefore \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

In the third order determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

if the rows and column are interchanged, it

becomes $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \Delta'$ (say)

If Δ is expanded by taking the constituents of the first column and Δ' is expanded by taking the constituents of the first row, then

$$\Delta = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

and $$\Delta' = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$\therefore \Delta = \Delta'$ (since the value of determinant of 2nd orders is unchanged if rows and columns are interchanged).

II. If two adjacent rows and columns of the determinant are interchanged the sign of the determinant is changed but its absolute value remains unaltered.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

A' has been obtained by interchanging the first and second rows of A

Expanding each determinant by the constituents of the first column.

$$\Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

and $A' = a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix}$

$$= -a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$\left[\text{since } \begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix} = b_2c_1 - c_2b_1 \text{ and } \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = b_1c_2 - b_2c_1\right] = -\Delta$$

In this way it can be proved that only the sign changes if any other two adjacent rows or columns are interchanged.

III. If two rows or columns of a determinant are identical, the determinant vanishes.

Let $\Delta_2 = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}$

The first two columns in the determinant are identical. If the first and second columns are interchanged, then the resulting determinant becomes $-A_2$ by II. But since these two columns are identical, the determinant remains unaltered by the interchange.

$\therefore A_2 = -A_2$ or, $2A_2 = 0$

$\therefore A_2 = 0$

IV. If each constitutent in any row or any column is multiplied by the same factor, then the determinant is multiplied by that factor.

Let $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

The determinant obtained when the constituents of the first row are multiplied by m is

$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = ma_1A_1 - ma_2A_2 + ma_3A_3$$

$= m\,[a_1A_1 - a_2A_2 + a_3A_3] = mA$

V. If each constituent in any row or column consists of two or more terms, then the determinant can be expressed as the sum of two or more than two other determinants in the determinant.

In the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let $a_1 = t_1 + m_1 + n_1$, $a_2 = t_2 + m_2 + n_2$, $a_3 = t_3 + m_3 + n_3$

Then the given determinant

$$= \begin{vmatrix} t_1 + m_1 + n_1 & b_1 & c_1 \\ t_2 + m_2 + n_2 & b_2 & c_2 \\ t_3 + m_3 + n_3 & b_3 & c_3 \end{vmatrix}$$

$= (t_1 + m_1 + n_1)\,A_1 - (t_2 + m_2 + n_2)\,A_2 + (t_3 + m_3 + n_3)\,A_3$

$= (t_1A_1 - t_2A_2 + t_3A_3) + (m_1A_1 - m_2A_2 + m_3A_3) + (n_1A_1 - n_2A_2 + n_3A_3)$

$$= \begin{vmatrix} t_1 & b_1 & c_1 \\ t_2 & b_2 & c_2 \\ t_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} n_1 & b_1 & c_1 \\ n_2 & b_2 & c_2 \\ n_3 & b_3 & c_3 \end{vmatrix}$$

It can be similarly proved that

$$= \begin{vmatrix} a_1 + p_1 & b_1 + q_1 & c_1 \\ a_2 + p_2 & b_2 + q_2 & c_2 \\ a_3 + p_3 & b_3 + q_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & q_1 & c_1 \\ a_2 & q_2 & c_2 \\ a_3 & q_3 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & b_1 & c_1 \\ p_2 & b_2 & c_2 \\ P_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 & c_1 \\ p_2 & q_2 & c_2 \\ p_3 & q_3 & c_3 \end{vmatrix}$$

VI. If the constituents of any row (or column) be increased or decreased by any equimultiples of the corresponding constituents of one or more of the other rows (or columns) the value of the determinant remains unaltered.

$$\text{Let } A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The determinant obtained, when the constituents of first column are increased by *l* times the second column *m* times the corresponding constituents of the third column is

$$\begin{vmatrix} a_1 + lb_1 + mc_1 & b_1 & c_1 \\ a_2 + lb_2 + mc_2 & b_2 & c_2 \\ a_3 + lb_3 + mc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} lb_1 & b_1 & c_1 \\ lb_2 & b_2 & c_2 \\ lb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mc_1 & b_1 & c_1 \\ mc_2 & b_2 & c_2 \\ mc_3 & b_3 & c_3 \end{vmatrix} \text{(by v)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + l\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + m\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} \text{(by iv)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta$$

SOLUTIONS OF SIMULTANEOUS LINEAR EQUATIONS

Cramer's Rule :

A method is given below for solving three simultaneous linear equations in three unknowns. This method may also be applied to solve ‘n’ equations in ‘n’ unknowns.

Consider the system of equations.

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad (1)$$

Where the coefficients are real.

The coefficient of x, y, z as noted in equations (1) may be used to form the determinant.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Which is called the determinant of the system.

If $\Delta \neq 0$, the solution of (1) is given by $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$, where Δ_r; r = 1, 2, 3 is the determinant obtained from Δ by replacing the r^{th} column by d_1, d_2, d_3.

Example –1 : Find the value of $\begin{vmatrix} \mathbf{5} & \mathbf{-2} & \mathbf{1} \\ \mathbf{3} & \mathbf{0} & \mathbf{2} \\ \mathbf{8} & \mathbf{1} & \mathbf{3} \end{vmatrix}$

Solution : The value of the given determinant

$$= 5\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 8 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & 0 \\ 8 & 1 \end{vmatrix}$$

$= 5\,(0 - 2) - 2\,(9 - 16) + 1\,(3 - 0)$

$= -10 + 14 + 3 = 7$

Example – 2. **Prove that** $\begin{vmatrix} \mathbf{a} & \mathbf{a^2} & \mathbf{a^3} \\ \mathbf{b} & \mathbf{b^2} & \mathbf{b^3} \\ \mathbf{c} & \mathbf{c^2} & \mathbf{c^3} \end{vmatrix}$ **= abc (a – b) (b – c) (c – a)**

Solution : L.H.S. $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$= abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ (Taking a, b, c, from R_1, R_2, R_3)

$= abc\begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$, replacing R_1 by $R_1 - R_2$ and R_2 by $R_2 - R_3$)

$= abc\,(a - b)\,(b - c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$ (Taking (a – b) & (b – c) common from R_1 & R_2 respectively)

$$= abc\,(a-b)\,(b-c)\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix} = abc\,(a-b)\,(b-c)\,(c-a)$$

Assignment

1. Find minors & cofactors of the determinants $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 2 \end{vmatrix}$

2. Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

3. Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

CHAPTER – 2

MATRIX

MATRIX AND ITS ORDER

INTRODUCTION :

In modern engineering mathematics matrix theory is used in various areas. It has special relationship with systems of linear equations which occour in many engineering processes.

A matrix is a reactangular array of numbers arranged in rows (horizontal lines) and columns (vertical lines). If there are 'm' rows and 'n' Column's in a matrix, it is called an 'm' by 'n' matrix or a matrix of order m × n. The first letter in mxn denotes the number of rows and the second letter 'n' denotes the number of columns. Generally the capital letters of the alphabet are used to denote matrices and the actual matrix is enclosed in parantheses.

$$\text{Hence } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & -- & a_{1n} \\ a_{21} & a_{22} & a_{23} & -- & a_{2n} \\ a_{31} & a_{32} & a_{33} & -- & a_{3n} \\ -- & -- & -- & -- & -- \\ a_{m1} & a_{m2} & a_{m3} & -- & a_{mn} \end{bmatrix}$$

is a matrix of order m × n and 'a_{ij}' denotes the element in the ith row and jth column. For example a_{23} is the element in the 2nd row and third column. Thus the matrix 'A' may be written as (a_{ij}) where i takes values from 1 to m to represent row and j takes values from 1 to n to represent column.

If m = n, the matrix A is called a square matrix of order n × n (or simply n). Thus

$$A = \begin{bmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & -- & a_{2n} \\ a_{31} & a_{32} & -- & a_{3n} \\ & & -- & \\ a_{n1} & a_{n2} & -- & --a_{nn} \end{bmatrix}$$

is a square matrix of order n. The determinant of order n, $\begin{vmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & -- & a_{2n} \\ a_{31} & a_{32} & -- & a_{3n} \\ & & -- & \\ a_{n1} & a_{n2} & -- & --a_{nn} \end{vmatrix}$

which is associated with the matrix 'A' is called the determinant of the matrix and is denoted by det A or |A|.

TYPES OF MATRICES WITH EXAMPLES

(a) **Row Matrix :** A matrix of order 1 × n is called a row matrix. For example (1 2), (a b c) are row matrices of order 1 × 2 and 1 × 3 respectively.

(b) **Column Matrix :** A matrix of order m × 1 is called a column matrix. The matrices $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}a\\b\end{bmatrix}$ are column matrices of order 3 × 1 and 2 × 1 respectively.

(c) **Zero matrix :** If all the elements of a matrix are zero it is called the zero matrix, (or null matrix) denoted by (0). The zero matrix may be of any order. Thus (0), (0, 0), $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ are all zero matrices.

(d) **Unit Matrix :** The square matrix whose elements on its main diagonal (left top to right bottom) are 1's and rest of its elements are 0's is called unit matrix. It is denoted by I and it may be of any order. Thus (1) $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ are unit matrices of order 1, 2, 3 respectively.

(e) **Singular and non -singular matrices :** A square matrix A is said to be singular if and only if its determinant is zero and is said to be non-singular (or regular) if det A $\neq$ 0.

For example $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ is a non singular matrix.

For $\begin{vmatrix}1 & 2\\3 & 4\end{vmatrix} = 4 - 6 = -2 \neq 0$ and $\begin{bmatrix}1 & 2 & 3\\3 & 4 & 5\\5 & 6 & 7\end{bmatrix}$ is a singular matrix

i.e. $\begin{vmatrix}1 & 2 & 3\\3 & 4 & 5\\5 & 6 & 7\end{vmatrix} = 0$

Adjoint of a Matrix :

The adjoint of a matrix A is the transpose of the matrix obtained replacing each element a_{ij} in A by its cofactor A_{ij}. The adjoint of A is written as adj A. Thus if

$$A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix} \text{ then adj } A = \begin{bmatrix}A_{11} & A_{21} & A_{31}\\A_{12} & A_{22} & A_{32}\\A_{13} & A_{23} & A_{33}\end{bmatrix}$$

Example – 1 : **Find inverse of the following matrices** $\begin{bmatrix}\mathbf{2} & \mathbf{-1}\\\mathbf{1} & \mathbf{3}\end{bmatrix}$

Sol^n : **(i)** Given A = $\begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}$, |A| = 7

$A^{-1} = \dfrac{\text{adj } A}{|A|}$, $|A| \neq 0$

So it has inverse

Adj (A)

Minor of 2, $M_{11} = 3$, Cofactor of 2, $C_{11} = 3$

Minor of –1, $M_{12} = 1$, Cofactor of –1, $C_{12} = -1$

Minoir of 1, $M_{21} = -1$, Cofactor of 1, $C_{21} = 1$

Minor of 3, $C_{22} = 2$, Cofactor of 3, $C_{22} = 2$

$$\text{adj}(A) = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}}{7} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix}$$

Assignment

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

   Calculate (i) AB (ii) BA

2. Find the inverse of the following : $\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$

❒ ❒ ❒

CO-ORDINATE GEOMETRY IN TWO DIMENSIONS

CHAPTER – 3

STRAIGHT LINE

CO-ORDINATE SYSTEM

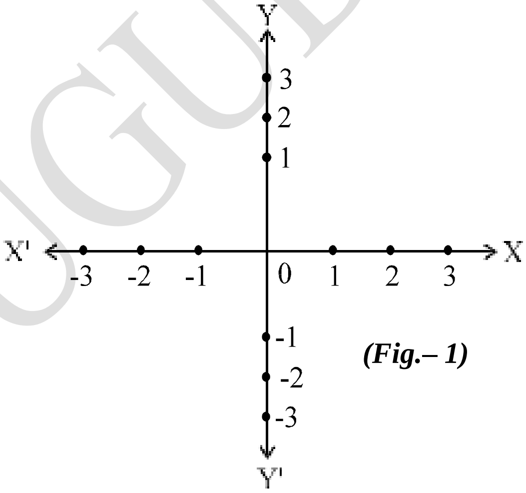
We represent each point in a plane by means of an ordered pair of real numbers, called co-ordinates. The branch of mathematics in which geometrical problems are solved through algebra by using the co-ordinate system, is known as co-ordinate geometry or analytical geometry.

Rectangular co-ordinate Axes

Let X'OX and YOY' be two mutually perpendicular lines (called co-ordinate axes), intersecting at the point O. ***(Fig.1)***.We call the point O, the origin, the horizontal line X'OX, the x-axis and the vertical line YOY', the y-axis.

We fix up a convenient unit of length and starting from the origin as zero, mark. distances on x-axis as well as y-axis.

The distance measured along OX and OY are taken as positive while those along OX' and OY' are considered negative.



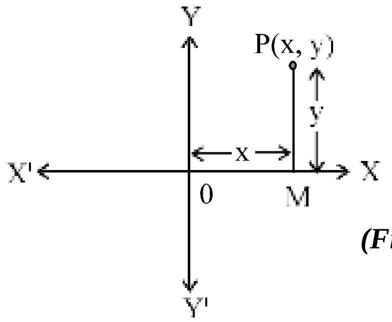
(Fig.– 1)

Cartesian co-ordinates of a point

Let X'OX and YOY' be the co-ordinate axes and let P be a point in the Euclidean plane ***(Fig.2)***. From P draw PM ⊥ X'OX.

Let OM = x and PM = y, Then the ordered pair (x, y) represents the cartesian co-ordinates of P and we denote the point by P(x, y). The number x is called the x-co-ordinate or abscissa of the point P, while y is known as its y-co-ordinate or ordinate.

Thus, for a given point the abscissa and the ordinate are the distances of the given point from y- axis and x-axis respectively.



(Fig.– 2)

Quadrants

The co-ordinate axes X'OX and Y'OY divide the plane in to four regions, called quadrants.

The regions XOY, YOX', X'OY' and Y'OX are known as the first, the second, the third and the fourth quadrant respectively. ***(Fig.3)***

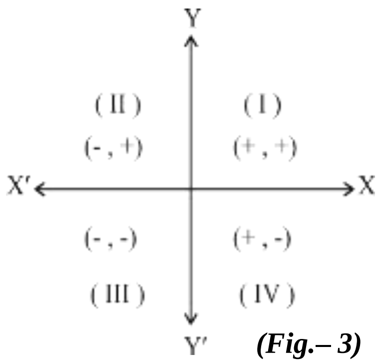
In accordance with the convention of signs defined above for a point (x, y) in different quadrants we have

1st quadrant : $x > 0$ and $y > 0$

2nd quadrant : $x < 0$ and $y > 0$

3rd quadrant : $x < 0$ and $y < 0$

4th quadrant : $x > 0$ and $y < 0$

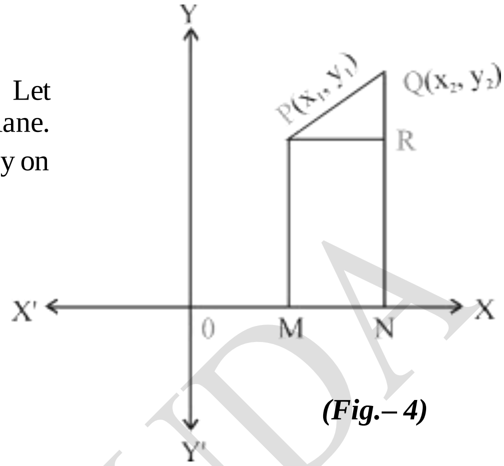


(Fig.– 3)

DISTANCE BETWEEN TWO GIVEN POINTS

The distance between any two points in the plane is the length of the line segment joining them.
The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



(Fig.– 4)

Proof : Let X'OX and YOY' be the co-ordinate axes ***(Fig.4)***. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two given points in the plane. From P and Q draw perpendicular PM and QN respectively on the x-axis. Also draw $PR \perp QN$.

Then, $OM = x_1$, $ON = x_2$

$PM = y_1$ & $QN = y_2$

$\therefore \quad PR = MN = ON - OM = x_2 - x_1$

and $QR = QN - RN = QN - PM = y_2 - y_1$

Now from right angled triangle PQR,

we have $PQ^2 = PR^2 + QR^2$ [by Pythagoras theorem]

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

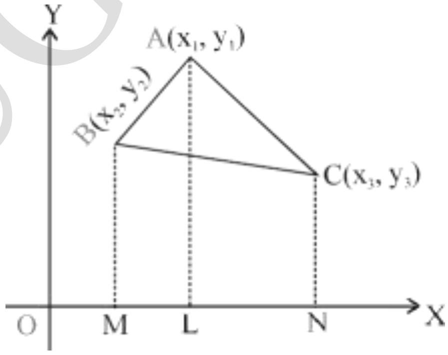
$\therefore |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Cor : The distance of a point P(x, y) from the origin O (0, 0) is

$$= \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$$

Area of a triangle :

Let ABC be a given triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. From the vertices A, B and C draw perpendiculars AL, BM and CN respectively on x-axis. ***(Fig.5)***.



(Fig.– 5)

Then, $ML = x_1 - x_2$; $LN = x_3 - x_1$ and $MN = x_3 - x_2$

$\therefore$ Area of Δ ABC

= area of trapezium ALMB + area of trapezium ALNC

– area of trapezium BMNC

$$= \frac{1}{2}(AL + BM) \,.\, ML + \frac{1}{2}(AL + CN) \,.\, LN - \frac{1}{2}(MB + CN) \,.\, MN$$

$$= \frac{1}{2}(y_1 + y_2)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}[x_1y_1 + x_1y_2 - x_2y_1 - x_2y_2 + x_3y_1 + x_3y_3 - x_1y_1 - x_1y_3 - x_3y_2 - x_3y_3 + x_2y_2 + x_2y_3]$$

$$= \frac{1}{2}[x_1y_2 - x_2y_1 + x_3y_1 - x_1y_3 - x_3y_2 + x_2y_3]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

In determinant form, we may write

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Condition for collinearity of Three points :

Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are colliner, i.e. lie on the same straight line, if the area of Δ ABC is zero. So the required condition for A, B, C to be collinear is that

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$$

$$\Rightarrow \quad x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

Formula for Internal Divisions :

The co-ordinates of a point P which divides the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio m : n are given by

$$\bar{x} = \frac{mx_2 + nx_1}{m+n},\ \bar{y} = \frac{my_2 + ny_1}{m+n}$$

Example – 1 : **In what ratio does the point (3, –2) divide the line segment joining the points (1, 4) and (–3, 16) :**

Solution : Let the point C (3, –2) divide the segment joining A(1, 4) and B (–3, 16) in the ratio K: 1

The co-ordinates of 'C' are $\left(\frac{-3k+1}{k+1}, \frac{16k+4}{k+1}\right)$

But we are given that the point C is (3, –2)

$\therefore$ We have $\frac{-3k+1}{k+1} = 3$

or $-3k + 1 = 3k + 3$

or $-6k = 2$

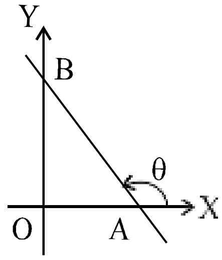
$\therefore k = -\frac{1}{3}$

$\therefore$ C divides AB in the ratio 1 : 3 externally.

SLOPE OF A LINE

Angle of Inclination : The angle of inclination or simply the inclination of a line is the angle θ made by the line with positive direction of x-axis, measured from it in anticlock wise direction ***(Fig. 6)***.

Slope or gradient of a line : If θ is the inclination of a line, then the value of tan θ is called the slope of the line and is denoted by m.



(Fig.– 6)

CONDITIONS OF PARALLELISM AND PERPENDICULARITY

1. **Two lines are parallel if and only if their slopes are equal.**
2. **Two lines with slope m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$**
3. **The slope of a line passing through two given points (x_1, y_1) and (x_2, y_2) is given by $m = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$**
4. **The equation of a line with slope m and making an intercept 'c' on y-axis is given by y = mx + c.**

Proof : Let AB be the given line with inclination θ so that $\tan\theta = m$. Let it intersect the y-axis at C so that OC = c. ***(Fig.7)***

Let it intersect the x-axis at A.

Let P(x, y) be any point on the line.

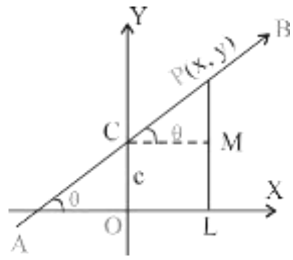
Draw PL perpendicular to x-axis and CM $\perp$ PL

Clearly, $\angle MCP = \angle OAC = \theta$

CM = OL = x ;

and PM = PL – ML = PL – OC = y – c

Now, from rt. angled Δ PMC



(Fig.– 7)

We get $\tan\theta = \dfrac{PM}{CM}$ or $m = \dfrac{y-c}{x}$

or $y = mx + c$, which is required equation of the line.

5. The equation of a line with slope m and passing through a point (x_1, y_1) is given by $(y - y_1) = m(x - x_1)$

6. The equation of a line through two given points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$$

7. The equation of a straight line which makes intercepts of length 'a' and 'b' on x-axis and y-axis respectively, is $\dfrac{x}{a} + \dfrac{y}{b} = 1$

Proof : Let AB be a given line meeting the x-axis and y-axis at A and B respectively ***(Fig.8)***.

Let OA = a and OB = b

Then the co-ordinates of A, B are A(a, 0) and B(0, b)

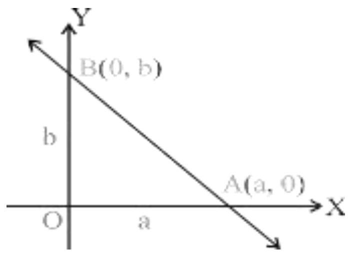
$\therefore$ The equation of the line joining A & B is

$$(y - 0) = \frac{b-0}{0-a}(x - a)$$

$$\Rightarrow \quad y = \frac{-b}{a}(x - a)$$

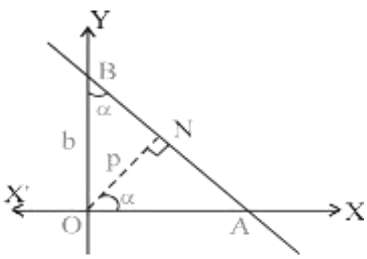
$$\Rightarrow \quad \frac{y}{b} = \frac{-x}{a} + 1$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$



(Fig.– 8)

8. Let P be the length of perpendicular from the origin to a given line and α be the angle made by this perpendicular with the positive direction of x-axis. Then the equation of the line is given by x cos a + y sin a = P



(Fig.– 9)

Conditions for two lines to be coincident, parallel, perpendicular or Intersect :

Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) conicident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$;

(ii) Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(iii) Perpendicular, if $a_1a_2 + b_1b_2 = 0$;

(iv) Intersecting, if they are neither coincident nor parallel.

Example – 1 : **Find the equation of the line which passes through the point (3, 4) and the sum of its intercept on the axes is 14.**

Solⁿ : Let the intercept made by the line on x-axis be 'a' and 'y'- axis be 'b'

i.e. $a + b = 14$ i.e, $b = 14 - a$

$\therefore$ Equation of the line is given by

$\frac{x}{a} + \frac{y}{14 - a} = 1$................(i)

As the point (3, 4) lies on it, we have

$\frac{3}{a} + \frac{4}{14 - a} = 1$

or $3(14 - a) + 4a = 14a - a^2$

or $42 - 3a + 4a = 14a - a^2$

or $a^2 - 13a + 42 = 0$

or $(a - 7)(a - 6) = 0$

or $a = 7$ or $a = 6$

Putting these values of a in (i)

$\frac{x}{7} + \frac{y}{7} = 1$ or $x + y = 7$

and $\frac{x}{6} + \frac{y}{8} = 1$ or $4x + 3y = 24$

Example – 2 : **Find the equation of the line passing through (–4, 2) and parallel to the line 4x – 3y = 0**

Solⁿ : Any line passing thorugh (–4, 2) whose equation is given by

$(y - 2) = m(x + 4)$........(i)

and parallel to the given line $4x - 3y = 0$

whose slope is $y = \frac{4}{3}x$

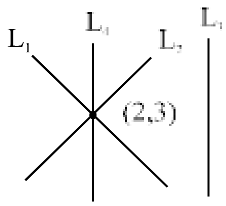
Here 'm' $= \frac{4}{3}$

It's equation is

$(y - 2) = \frac{4}{3}(x + 4)$

$3y - 6 = 4x + 16$

or $4x - 3y + 22 = 0$



(Fig.– 10)

Example – 3 : **Find the equation of the line passing through the intersection of 2x – y – 1 = 0 and 3x – 4y + 6 = 0 and parallel to the line x + y – 2 = 0**

Sol^n : Point of intersection of $2x - y - 1 = 0$ and $3x - 4y + 6 = 0$

$$\left(\frac{-1\times 6-(-4)(-1)}{2(-4)-3(-1)}, \frac{(-1)\times 3-6(2)}{2(-4)-3(-1)}\right)$$

$$= \left(\frac{-6-4}{-8+3}, \frac{-3-12}{-8+3}\right) = \left(\frac{-10}{-5}, \frac{-15}{-5}\right) = (2, 3)$$

Any line parallel to the line $x + y - 2$ is given by $x + y + k = 0$. ...(i)

Since the line passes through (2, 3) hence it satisfies the equation (i)

So, $2 + 3 + k = 0$

$\Rightarrow k = -5$

Now putting the value of k in equation (i), we get $x + y - 5 = 0$

$\therefore$ Equation of the line is $x + y - 5 = 0$

Assignment

1. Find the equation of a line parallel to $2x + 4y - 9 = 0$ and passing through the point $(-2, 4)$
2. Find the co-ordinates of the foot of the perpendicular from the point (2, 3) on the line $3x - 4y + 7 = 0$
3. Find the equation of the line through the point of intersection of $3x + 4y - 7 = 0$ and $x - y + 2 = 0$ and which is parallel to the line $5x - y + 11 = 0$

CHAPTER – 4

CIRCLE

A circle is the locus of a point which moves in a plane in such a way that it's distance from a fixed point is always constant.

The fixed point is called the centre of the circle and the constant distance is called its radius.

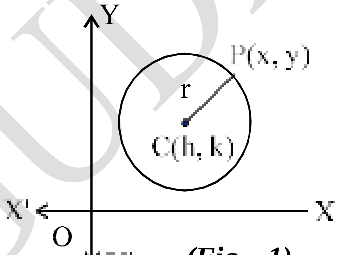
Equation of a circle (Standard form)

Let C(h, k) be the centre of a circle with radius 'r' and let P (x, y) be any point on the circle ***(Fig.1)***.

Then $CP = r \rightarrow CP^2 = r^2$

$\rightarrow (x-h)^2 + (y-k)^2 = r^2$

Which is required equation of the circle.



(Fig.– 1)

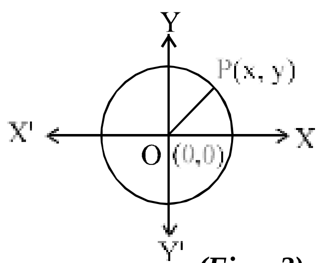
Cor. The equation of a circle with the centre at the origin and radius r, is $x^2 + y^2 = r^2$ ***(Fig.2)***.

Proof : Let O (0, 0) be the centre and r be the radius of a circle and let P (x, y) be any point on the circle.

Then $OP = r \rightarrow OP^2 = r^2$

$\rightarrow (x-0)^2 + (y-0)^2 = r^2$

$\rightarrow x^2 + y^2 = r^2$



(Fig.– 2)

Example – 1. **Find the equation of a circle with centre (–3, 2) and radius 7.**

Sol^n **:** The required equation of the circle is

$[x-(-3)]^2 + (y-2)^2 = 7^2$

or $(x+3)^2 + (y-2)^2 = 49$

or $x^2 + y^2 + 6x - 4y - 36 = 0$

Example – 2. **Find the equation of a circle whose centre is (2, –1) and which passes through (3, 6)**

Sol^n **:** Since the point P (3, 6) lies on the circle, its distance from the centre C (2, –1) is therefore equal to the radius of the circle.

$\therefore$ Radius = CP = $\sqrt{(3-2)^2 + (6+1)^2} = \sqrt{50}$

So, the required equation of the circle is

$(x-2)^2 + (y+1)^2 = 50$

or $x^2 + y^2 - 4x + 2y - 45 = 0$

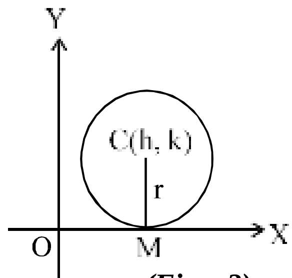
Example – 3. **Find the equation of a circle with centre (h, k) and touching the x-axis *(Fig.3)*.**

Sol^n **:** Clearly, the radius of the circle = CM = r = k

So, the required equation

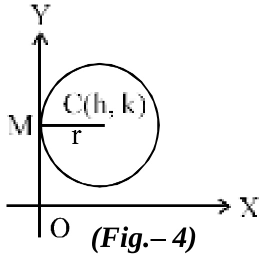
$(x-h)^2 + (y-k)^2 = k^2$

or $x^2 + y^2 - 2hx - 2ky + h^2 = 0$



(Fig.– 3)

***Example – 4.* Find the equation of a circle with centre (h,k) and touching y-axis*(Fig.4)*.**



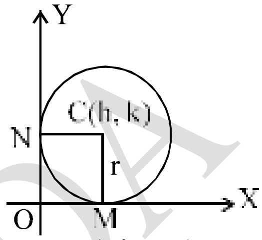
(Fig.– 4)

***Solⁿ* :** Clearly, the radius of the circle = CM = r = h

So, the required equation is $(x – h)^2 + (y – k)^2 = h^2$

or $x^2 + y^2 – 2hx – 2ky + k^2 = 0$

***Example – 5.* Find the equation of a circle with centre (h,k) and touching both the axes *(Fig.5)*.**



(Fig.– 5)

***Solⁿ* :** Clearly, radius, CM = CN = r

i.e. h = k = r (say)

$\therefore$ the equation of the circle is $(x – r)^2 + (y – r)^2 = r^2$

or $x^2 + y^2 – 2r (x + y) + r^2 = 0$

GENERAL EQUATION OF A CIRCLE

Theorem : The general equation of a circle is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$
And, every such equation represents a circle.

Proof : The standard equation of a circle with centre (h, k) and radius r is given by

$(x – h)^2 + (y – k)^2 = r^2$

Or $x^2 + y^2 – 2hx – 2ky + (h^2 + k^2 – r^2) = 0$

This is of the form

$x^2 + y^2 + 2gx + 2fy + c = 0$

Where $h = –g, k = –f$ and $c = (h^2 + k^2 – r^2)$

Conversely, let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the given condition.

Then, $x^2 + y^2 + 2gx + 2fy + c = 0$

$\rightarrow (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = (g^2 + f^2 – c)$

$\rightarrow (x + g)^2 + (y + f)^2 = \left(\sqrt{g^2 + f^2 – c}\right)^2$

$\rightarrow [x – (–g)]^2 + [y – (–f)]^2 = \left[\sqrt{g^2 + f^2 – c}\right]^2$

$\rightarrow (x – h)^2 + (y – k)^2 = r^2$

Where $h = – g$, $k = –f$ and $r = \sqrt{g^2 + f^2 – c}$

This shows that the given equation represents a circle with centre (–g, –f) and radius.

$= \sqrt{g^2 + f^2 – c}$, provided $g^2 + f^2 > c$.

EQUATION OF A CIRCLE WITH GIVEN END POINTS OF A DIAMETER

Theorem : The equation of a circle described on the line joining the points $A(x_1, y_1)$ and $B (x_2, y_1)$ as a diameter, is $(x – x_1)(x – x_2) + (y – y_1) (y – y_2) = 0$

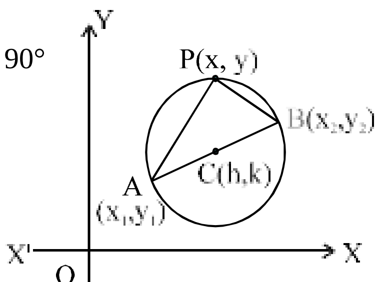
Proof : Let $A (x_1, y_1)$ and $B (x_2, y_2)$ be the end point of a diameter of the given circle and let P (x, y) be any point on the circle *(Fig.6)*.

Since the angle in a semi-circle is a right angle, we have $\angle APB = 90°$

Now slope of AP = $\left(\dfrac{y – y_1}{x – x_1}\right)$

And, slope of BP = $\left(\dfrac{y – y_2}{x – x_2}\right)$

Since AP $\perp$ BP, we have



(Fig.– 6)

$$\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_1}\right) = -1$$

Or $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

***Example – 1.* Find the equation of a circle whose end points of diameter are (3, 4) and (–3, –4)**

***Sol^n*. :** The required equation of the circle is $(x-3)(x+3) + (y-4)(y+4) = 0$

i.e. $x^2 - 9 + y^2 - 16 = 0$

or $x^2 + y^2 = 25$

***Example – 2.* Find the centre and radius of the circle.**

$$\mathbf{x^2 + y^2 - 6x + 4y - 36 = 0}$$

***Sol^n*. :** Comparing the equation with

$x^2 + y^2 + 2gx + 2fy + c = 0$

We get $2g = -6$, $2f = 4$ and $c = -36$

or $g = -3$, $f = 2$ and $c = -36$

$\therefore$ Centre of the circle is $(-g, -f)$, i.e. $(3, -2)$

And radius of the circle.

$= \sqrt{g^2 + f^2 - c} = \sqrt{9 + 4 + 36} = 7$

Assignment

1. Find the centre and radius of each of the following circles
   $x^2 + y^2 + x - y - 4 = 0$
2. Find the equation of the circle whose centre is (–2, 3) and passing through origin
4. Find the equation of the circle having centre at (1, 4) and passing through (–2, 1).
4. Find the equation of the circle passing through the points (1, 3) (2, –1) and (–1, 1).

TRIGONOMETRY

CHAPTER – 5

COMPOUND ANGLES

INTRODUCTION :

The word Trigonometry is derived from Greek words ‘‘Trigonos’’ and metrons means measurement of angles in a triangle. This subject was originally devecpaed to solve geometric problems involving trigangles. The Hindu mathematicians Aryabhatta, Varahmira, Bramhaguptu and Bhaskar have lot of contaribution to trigonometry . Besides Hindu mathematicians ancient-Greek and Arwric mathematicians also contributed a lot to this subject. Trigonometry is used in many are as such as science of seismology, designing electrical circuits, analysing musical tones and studying the occurance of sun spots.

Trigonometric Functions :

Let θ be the meausre of any angle measured in radians in counter clockmise sense as show in Fig (1).

Let P(x, y) be any point an the terminal side of angle θ. The distance of P from O is OP = r = $\sqrt{x^2+y^2}$. the functions defined by $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$ ...(1) are called sine, cosine and tangent functions respectirely. These are called trigonometric functions. It followrs from (1) that $\sin^2\theta + \cos^2\theta = 1$. Other trigonomatric functions such as cosecant, secant and cotangent functions are defined as $\operatorname{cosec}\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$, $\cot\theta = \frac{x}{y}$.

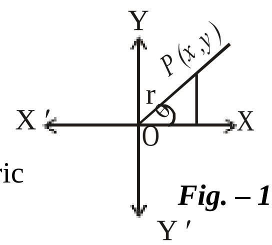
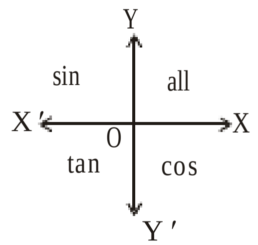


Fig. – 1

SIGN OF T-RATIOS :

The student may remember the signs of t-ratios in different quadrant with the help of the diagram



The sign of paricular t-ratio in any quadrant can be remembered by the word ‘‘all-sin-tan-cos’’ or ‘‘add sugar to coffee’’. What ever is written in a particular quadrant along with its reciprocal is +ve and the rest are negetive.

Table giving the values of trigonometrical Ratios of angles 0º, 30°, 45º, 60º & 90º

θ	0º	30º	45º	60º	90º
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

RELATED ANGLES :

Definitions : Two angles are said to be complementary angles if their sum is 90° and each angle is said to be the complement of the other.

Two angles are said to be supplementary if their sum is 180° and each angles is said to be the supplement of the other.

To Find the T-Ratios of angle ($-\theta$) in terms of θ :

Let OX be the initial line. Let OP be the position of the radius vector after tracing an angle θ in the anticlockwise sense which we take as positive sense. ***(Fig. 2)***

Let OP' be the position of the radius vector after tracing (θ) in the clockwise sense, which we take as negative sense. So $\angle$ P' OX will be taken as $-\theta$. Join PP'. Let it meet OX at M.

Now Δ OPM $\equiv$ Δ P'OM, $\angle$ P' OM = $-\theta$

OP' = OP, P'M = – PM

$$\text{Now } \sin(-\theta) = \frac{P'M}{OP'} = \frac{-PM}{OP} = -\sin\theta$$

$$\cos(-\theta) = \frac{OM}{OP'} = \frac{OM}{OP} = \cos\theta$$

$$\tan(-\theta) = \frac{P'M}{OM} = \frac{-PM}{OM} = -\tan\theta$$

$$\operatorname{cosec}(-\theta) = \frac{OP'}{P'M} = \frac{OP}{-PM} = -\operatorname{cosec}\theta$$

$$\sec(-\theta) = \frac{OP'}{OM} = \frac{OP}{OM} = \sec\theta$$

$$\cot(-\theta) = \frac{OM}{P'M} = \frac{OM}{-PM} = -\cot\theta$$

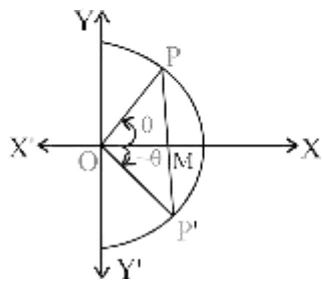


Fig. – 2

To find the T-Ratios of angle (90° – θ) in terms of θ.

Let OPM be a right angled triangle with $\angle$ POM = 90°, $\angle$ OMP = θ ,

$\angle$ OPM = 90°– θ . ***(Fig. 3)***

$$\therefore \quad \sin(90° - \theta) = \frac{OM}{PM} = \cos\theta \quad \Rightarrow \operatorname{cosec}(90° - \theta) = \sec\theta$$

$$\cos(90° - \theta) = \frac{OP}{PM} = \sin\theta \quad \Rightarrow \quad \sec(90° - \theta) = \operatorname{cosec}\theta$$

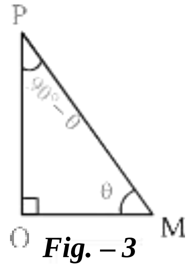


Fig. – 3

$$\tan(90° - \theta) = \frac{OM}{OP} = \cot\theta \quad \Rightarrow \cot(90° - \theta) = \tan\theta$$

To find the T-Ratios of angle (90° + θ) in terms of θ.

Let ∠ POX = θ and ∠ P'OX = 90° + θ . Draw PM and P'M' perpendiculars to the X-axis ***(Fig. 4)***

Now Δ POM ≅ Δ P'OM'

∴ P'M' = OM and OM' = –PM

$$\text{Now } \sin(90° + \theta) = \frac{P'M'}{OP'} = \frac{OM}{OP} = \cos\theta$$

$$\cos(90° + \theta) = \frac{OM'}{OP'} = \frac{-PM}{OP} = -\sin\theta$$

$$\tan(90° + \theta) = \frac{P'M'}{OM'} = \frac{OM}{-PM} = -\cot\theta$$

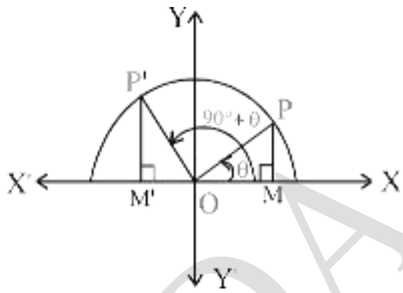


Fig. – 4

Similarly cosec (90° + θ) = sec θ

sec (90° + θ) = – cosec θ

and cot (90° + θ) = – tan θ

To Find the T-Ratios of angle (180° – θ) in terms of θ.

Let OX be the initial line. Let OP be the position of the radius vector after tracing an angle XOP = θ To obtain the angle 180° – θ let the radius vector start from OX and after revolving through 180° come to the position OX'. Let it revolve back through an angle θ in the clockwise direction and come to the position OP' so that the angle X'OP' is equal in magnitude but opposite in sign to the angle XOP. The angle XOP' is 180° – θ. ***(Fig.5)***

Draw P'M' and PM perpendicular to X'OX.

Now Δ POM ≡ Δ P'OM'.

∴ OM' = –OM and P'M' = PM

$$\text{Now } \sin(180° - \theta) = \frac{P'M'}{OP'} = \frac{PM}{OP} = \sin\theta$$

$$\cos(180° - \theta) = \frac{OM'}{OP'} = -\frac{OM}{OP} = -\cos\theta$$

$$\tan(180° - \theta) = \frac{P'M'}{OM'} = \frac{PM}{-OM} = -\tan\theta$$

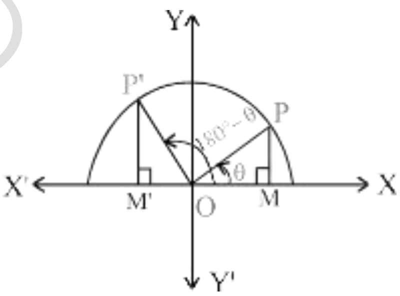


Fig. – 5

Similarly cosec (180° – θ) = cosec θ

sec (180° – θ) = –sec θ

and cot (180° – θ) = – cot θ

To Find the T-Ratios of angle (180° + θ) in terms of θ.

Let ∠ POX = θ and ∠ P'OX = 90° + θ .***(Fig. 6)***

Now Δ POM ≡ Δ P'OM'.

∴ OM' = –OM and P'M' = –PM

$$\text{Now } \sin(180° + \theta) = \frac{P'M'}{OP'} = \frac{-PM}{OP} = -\sin\theta$$

$$\cos(180° + \theta) = \frac{OM'}{OP'} = -\frac{OM}{OP} = -\cos\theta$$

$$\tan(180° + \theta) = \frac{P'M'}{OM'} = \frac{-PM}{-OM} = \tan\theta$$

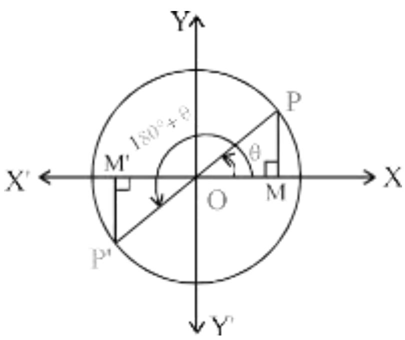


Fig. – 6

Similarly cosec $(180° + \theta)$ = cosec θ

sec $(180° + \theta) = -\sec\theta$

and cot $(180° + \theta) = \cot\theta$.

To Find the T-Ratios of angles $(270° \pm \theta)$ in terms of θ.

The trigonometrical ratios of $270° - \theta$ and $270° + \theta$ in terms of those of θ, can be deduced from the above articles. For example

$\sin(270° - \theta) = \sin[180° + (90° - \theta)]$

$= -\sin(90° - \theta) = -\cos\theta$

$\cos(270° - \theta) = \cos[180° + (90° - \theta)]$

$= -\cos(90° - \theta) = -\sin\theta$

Similarly $\sin(270° + \theta) = \sin[180° + (90° + \theta)]$

$= -\sin(90° + \theta) = -\cos\theta$

$\cos(270° + \theta) = \cos[180° + (90° + \theta)]$

$= -\cos(90° + \theta) = -(-\sin\theta) = \sin\theta$

To Find the T-Ratios of angles $(360° \pm \theta)$ in terms of θ.

We have seen that if n is any integer, the angle $n.360° \pm \theta$ is represented by the same position of the radius vector as the angle $\pm\theta$. Hence the trigonometrical ratios of $360° \pm\theta$ are the same as those of $\pm\theta$.

Thus $\sin(n.360° + \theta) = \sin\theta$

$\cos(n.360° + \theta) = \cos\theta$

$\sin(n.360° - \theta) = \sin(-\theta) = -\sin\theta$

and $\cos(n.360° - \theta) = \cos(-\theta) = \cos\theta$.

Examples :

$\cos(-720° - \theta) = \cos(-2 \times 360° - \theta) = \cos(-\theta) = \cos\theta$

and $\tan(1440° + \theta) = \tan(4 \times 360° + \theta) = \tan\theta$

In general when is any integer, $n \in Z$

(1) $\sin(n\pi + \theta) = (-1)^n \sin\theta$

(2) $\cos(n\pi + \theta) = (-1)^n \cos\theta$

(3) $\tan(n\pi + \theta) = \tan\theta$ when n is odd integer

(4) $\sin\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}} \cos\theta$

(5) $\cos\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n+1}{2}} \sin\theta$

(6) $\tan\left(\frac{n\pi}{2} + \theta\right) = \cot\theta$

EVEN FUNCTION :

A function $f(x)$ is said to be an even function of x, if $f(x)$ satisfies the relation $f(-x) = f(x)$.

Ex. cosx, secx, and all even powers of x i.e, x^2, x^4, x^6.......... are even function.

ODD FUNCTION :

A function $f(x)$ is said to be an odd function of x, if $f(x)$ satisfies the relation $f(-x) = -f(x)$.

Ex. sinx, cosec x, tan x, cot x and all odd powers of x i.e, x^3, x^5, x^7...... are odd function.

Example : *Find the values of sin*8 *and tan*8 *if cos*8 = $\frac{-12}{13}$ *and* 8 *lies in the third quadrant.*

Solution : We have $\sin^2 0 + \cos^2 0 = 1$

$$\Rightarrow \sin\theta = \sqrt{1-\cos^2\theta}$$

In third quadrant sin0 is negetive, therefore

$$\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\left(\frac{-12}{13}\right)^2} = \frac{-5}{3}$$

$$\text{Now } \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-5}{13} \times \frac{13}{-12} = \frac{5}{12}$$

Example : Find the values of

(i) tan (–900º) **(ii) sin 1230º**

Solution : (i) $\tan(-900^\circ) = -\tan 900^\circ = -\tan(10 \times 90^\circ + 0^\circ) = -\tan 0^\circ = 0$

(ii) $\sin(1230^\circ) = \sin(6 \times 180^\circ + 150^\circ) = \sin 150^\circ = \sin(180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}$

Example : Show that

$$\frac{\cos(90^\circ+0).\sec(-0).\tan(180^\circ-0)}{\sec(360^\circ-0).\sin(180^\circ+0)\,.\cot(90^\circ-0)} = -1 = \frac{-\sin 0 \times \sec 0 \times -\tan 0}{\sec 0 \times -\sin 0 \times \tan 0} = -1$$

Solution : $$\frac{\cos(90^\circ+\theta).\sec(-\theta).\tan(180^\circ-\theta)}{\sec(360^\circ-\theta).\sin(180^\circ+\theta)\,.\cot(90^\circ-\theta)} = \frac{-\sin\theta \times \sec\theta \times -\tan\theta}{\sec\theta \times -\sin\theta \times \tan\theta} = -1$$

ASSIGNMENT

1. Find the value of cos1º.cos2º..... cos 100º

2. Evaluale : $\tan\frac{\pi}{20} \cdot \tan\frac{3\pi}{20} \cdot \tan\frac{5\pi}{20} \cdot \tan\frac{7\pi}{20} \cdot \tan\frac{9\pi}{20}$ ·

COMPOUND, MULTIPLE AND SUB-MULTIPLE ANGLES

When an angle formed as the algebric sum of two or more angles is called a compound angles. Thus A + B and A + B + c are compound angles.

Addition Formulae

When an angle formed as the algebraical sum of two or more angles, it is called a compound angles. Thus A + B and A + B + C are compound angles.

Addition Formula :

(i) $\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

(ii) $\cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

(iii) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

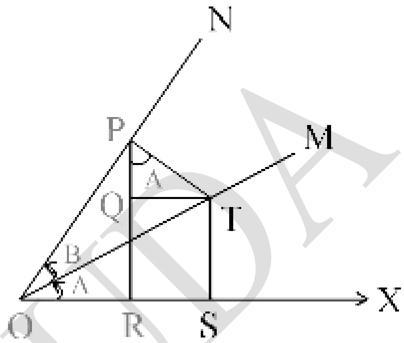


Fig. – 7

Proof : Let the revolving line OM starting from the line OX make an angle XOM = A and then further move to make.

$\angle MON = B$, so that $\angle XON = A + B$ ***(Fig. 7)***

Let 'P' be any point on the line ON.

Draw $PR \perp OX$, $PT \perp OM$, $TQ \perp PR$ and $TS \perp OX$

Then $\angle QPT = 90° - \angle PTQ = \angle QTO = \angle XOM = A$

$\therefore$ We have from A OPR

(i) $\sin(A + B) = \dfrac{RP}{OP} = \dfrac{QR + PQ}{OP} = \dfrac{TS + PQ}{OP}$ (Q QR = TS)

$= \dfrac{TS}{OP} + \dfrac{PQ}{OP} = \dfrac{TS}{OT} \cdot \dfrac{OT}{OP} + \dfrac{PQ}{PT} \cdot \dfrac{PT}{OP}$

$= \sin A \cdot \cos B + \cos A \cdot \sin B$

(ii) $\cos(A + B) = \dfrac{OR}{OP} = \dfrac{OS - RS}{OP} = \dfrac{OS}{OP} - \dfrac{RS}{OP}$

$= \dfrac{OS}{OP} - \dfrac{QT}{OP}$ [Q RS = QT]

$= \dfrac{OS}{OT} \cdot \dfrac{OT}{OP} - \dfrac{QT}{PT} \cdot \dfrac{PT}{OP}$

$= \cos A \cdot \cos B - \sin A \cdot \sin B$

(iii) $\tan(A + B) = \dfrac{\sin(A + B)}{\cos(A + B)}$

$= \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B)}$

(dividing numerator and denominator by cos A cos B)

$$= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

(iv) $\cot (A + B) = \dfrac{\cos(A + B)}{\sin(A + B)}$

$= \dfrac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$

[dividing of numerator and denominator by sin A sin B]

$= \dfrac{\dfrac{\cos A \cos B}{\sin A \sin B} - 1}{\dfrac{\sin A \cos B}{\sin A \sin B} + \dfrac{\cos A \sin B}{\sin A \sin B}}$

$\cot (A + B) = \dfrac{\cot A . \cot B - 1}{\cot B + \cot A}$

Cor : In the above formulae, replacing A by $\dfrac{\pi}{2}$ and B by x

We have

$\sin\left(\dfrac{\pi}{2} + x\right) = \sin\dfrac{\pi}{2} . \cos x + \cos\dfrac{\pi}{2} . \sin x$

$= 1 . \cos x + 0 . \sin x = \cos x$

$\cos\left(\dfrac{\pi}{2} + x\right) = \cos\dfrac{\pi}{2} . \cos x - \sin\dfrac{\pi}{2} . \sin x$

$= 0 \times \cos x - 1 \times \sin x = -\sin x$

$\tan\left(\dfrac{\pi}{2} + x\right) = \dfrac{\sin\left(\dfrac{\pi}{2} + x\right)}{\cos\left(\dfrac{\pi}{2} + x\right)} = \dfrac{\cos x}{-\sin x} = -\cot x$

(b) Difference Formulae :

(i) $\sin (A - B) = \sin A . \cos B - \cos A . \sin B$

(ii) $\cos (A - B) = \cos A . \cos B + \sin A . \sin B$

(iii) $\tan (A - B) = \dfrac{\tan A - \tan B}{1 + \tan A . \tan B}$

Proof : Let the reveolving line OM make an angle A with OX and then resolve back to make $\angle$ MON = B so that $\angle$ XON = A – B. ***(Fig. 8)***

Let 'P' be any point on ON. Draw PR $\perp$ OX ,

PT $\perp$ OM , TS $\perp$ OX , TQ $\perp$ RP produced to Q.

Then $\angle$ TPQ = 90° – $\angle$ PTQ = $\angle$ QTM = A

Now from Δ OPR, we have

(i) $\text{Sin}(A - B) = \dfrac{PR}{OP} = \dfrac{QR - QP}{OP} = \dfrac{TS - QP}{OP}$

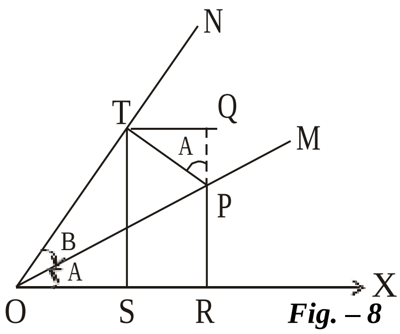


Fig. – 8

$= \frac{TS}{OP} - \frac{QP}{OP}$

$= \frac{TS}{OT} \cdot \frac{OT}{OP} - \frac{OP}{PT} \cdot \frac{PT}{OP}$

= sin A . cos B – cos A . sin B

(ii) $\cos(A-B) = \frac{OR}{OP} = \frac{OS+SR}{OP} = \frac{OS+TQ}{OP} = \frac{OS}{OP} + \frac{TQ}{QP}$

$= \frac{OS}{OT} \cdot \frac{OT}{OP} + \frac{TQ}{PT} \cdot \frac{PT}{OP}$

= cos A . cos B + sin A . sin B

(iii) $\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A.\cos B - \cos A.\sin B}{\cos A \cos B + \sin A \sin B}$

$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Dividing the numerator and the denominator by cos A. cos B.

(iv) $\cot(A-B) = \frac{\cos(A-B)}{\sin(A-B)}$

$= \frac{\cos A.\cos B + \sin A.\sin B}{\sin A.\cos B - \cos A.\sin B}$

$= \frac{\cot A.\cot B + 1}{\cot B - \cot A}$

dividing the numerator and denominator by sin A. sin B

We can also deduce substraction formulae from addition formulae in the following manner.

sin(A – B) = sin[A + (–B)]

= sin A . cos (–B) + cos A . sin (–B)

= sin A . cos B + cos A . sin B

cos(A – B) = cos[A + (–B)]

= cos A . cos (–B) – sin A . sin (–B)

= cos A . cos B + sin A . sin B

$\tan(A-B) = \tan[A+(-B)] = \frac{\tan A + \tan(-B)}{1 - \tan A . \tan(-B)} = \frac{\tan A - \tan B}{1 + \tan A . \tan B}$

***Example – 1 :* Find the value of tan 75° and hence prove that tan 75° + cot 75° = 4**

Solution: $\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1\times 1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$

$\therefore \quad \tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$\therefore \quad \cot 75° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} \quad \left[\text{since } \cot\theta = \dfrac{1}{\tan\theta}\right]$

$\tan 75° + \cot 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} + \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = \dfrac{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)}$

$= \dfrac{3+1+2\sqrt{3}+3+1-2\sqrt{3}}{3-1} \qquad [\text{since } (a+b)(a-b) = a^2 - b^2]$

$\therefore \quad \tan 75° + \cot 75° = 4$

***Example – 2 :* If sin A = $\dfrac{1}{\sqrt{10}}$ and sin B = $\dfrac{1}{\sqrt{5}}$ show that A + B = $\dfrac{\pi}{4}$**

Solution: sin A = $\dfrac{1}{\sqrt{10}}$

$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\dfrac{1}{10}} = \sqrt{\dfrac{10-1}{10}} = \sqrt{\dfrac{9}{10}}$

$\therefore \cos A = \dfrac{3}{\sqrt{10}}$

$\sin B = \dfrac{1}{\sqrt{5}}, \ \cos B = \sqrt{1-\sin^2 B}$

$= \sqrt{1-\dfrac{1}{5}} = \sqrt{\dfrac{5-1}{5}} = \sqrt{\dfrac{4}{5}}$

$\therefore \cos B = \dfrac{2}{\sqrt{5}}$

Sin(A + B) = sin A cos B + cos A sin B

$= \dfrac{1}{\sqrt{10}} \times \dfrac{2}{\sqrt{5}} + \dfrac{3}{\sqrt{10}} \times \dfrac{1}{\sqrt{5}} = \dfrac{2}{\sqrt{50}} + \dfrac{3}{\sqrt{50}}$

$= \dfrac{2+3}{\sqrt{50}} = \dfrac{2+3}{5\sqrt{2}}$

$\therefore \ \sin(A+B) = \dfrac{5}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

sin (A + B) = sin 45°

$\therefore A + B = 45° = \dfrac{\pi}{4} \quad \left[\text{since } 45° = \dfrac{180°}{4}\right]$

Transformation of Sums or Difference in to Products

(a) We have that

sin (A + B) + sin (A – B) = 2 sin A cos B ....(1)

sin (A + B) – sin (A – B) = 2 cos A sin B ....(2)

cos (A + B) – cos (A – B) = 2 cos A cos B ....(3)

cos (A – B) – cos (A + B) = 2 sin A sin B.........(4)

Let A + B = C and A – B = D

Then A = $\dfrac{C+D}{2}$ and B = $\dfrac{C-D}{2}$

Putting the value in formula (1), (2), (3) and (4) we get

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2} \quad \text{.................. (i)}$$

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2} \quad \text{.................. (ii)}$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2} \quad \text{................ (iii)}$$

$$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2} \quad \text{................ (iv)}$$

for practice it is more convenient to quote the formulae verbally as follows :

Sum of two sines = 2 sin (half sum) cos (half difference)

Difference of two sines = 2 cos (half sum) sin (half difference)

Sum of two cosines = 2 cos (half sum) cos (half difference)

Difference of two cosines = 2 sin (half sum) sin (half difference reversed)

[The student should carefully notice that the second factor of the right hand member of IV is sin $\frac{D-C}{2}$, not sin $\frac{C-D}{2}$]

(b) To find the Trigonometrical ratios of Angle 2A in terms of those of A : sin 2A, cos 2A.

Since sin (A + B) = sin A cos B + cos A sin B

putting B = A

sin (A + A) = sin A cos A + cos A sin A

$\Rightarrow$ sin 2A = 2 sin A cos A ....(i)

cos (A + B) = cos A cos B – sin A sin B

$\Rightarrow$ cos (A + A) = cos A cos A – sin A sin A

$\Rightarrow$ cos 2A = $\cos^2 A - \sin^2 A$..............................(ii)

Also cos 2A = $1 - \sin^2 A - \sin^2 A = 1 - 2\sin^2 A$ ....(iii)

So $2\sin^2 A = 1 - \cos 2A$.. (iv)

Also cos 2A = $\cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$............... (v)

or $2\cos^2 A = 1 + \cos 2A$ .... (vi)

(c) Formula for tan 2A

since $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\tan 2A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$= \frac{2 \tan A}{1 - \tan^2 A}$$

Note this formula is not defined when $\tan^2 A = 1$ i.e, $\tan A = \pm 1$

(d) To express sin 2A and cos 2A in terms of tan A

sin 2A = 2 sin A cos A

$$= 2\frac{\frac{\sin A}{\cos A}}{\frac{1}{\cos^2 A}} = \frac{2 \tan A}{\sec^2 A} = \frac{2 \tan A}{1 - \tan^2 A}$$

Also, $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\sin^2 A}{\cos^2 A}} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(dividing numerator and denominator by $\cos^2 A$)

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(e) To find the Trigonometrical formulae of 3A

$\sin 3A = \sin (2A + A)$

$= \sin 2A \cos A + \cos 2A \sin A$

$= 2 \sin A \cos A . \cos A + (1 - 2 \sin^2 A) \sin A$

$= 2 \sin A(1 - \sin^2 A) + (1 - 2 \sin^2 A) \sin A$

$= 3 \sin A - 4 \sin^3 A$

Again, $\cos 3A = \cos (2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A. \sin A$

$= (2 \cos^2 A - 1) \cos A - 2 \cos A (1 - \cos^2 A)$

$= 4 \cos^3 A - 3 \cos A$

Also $\tan 3A = \tan (2A + A)$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$= \frac{\frac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2 \tan A}{1 - \tan^2 A} . \tan A}$$

$$= \frac{2 \tan A + \tan A(1 - \tan^2 A)}{1 - \tan^2 A - 2 \tan^2 A}$$

$$= \frac{3 \tan A - \tan^3 A}{1 - 3\tan^2 A}$$, provided $3 \tan^2 A \neq 1$ i.e, $\tan A \neq \pm \frac{1}{\sqrt{3}}$

(f) Submultiple Angles :

To express trigonometric ratios of A in terms of ratios of A/2

sin 20 = 2 sin 0 cos 0 (true for all value of 0)

Let 20 = A i.e. $0 = \frac{A}{2}$

$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$ .... (i)

$\cos 20 = \cos^2 0 - \sin^2 0$

or $\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$ .... (ii)

$\cos A = 2 \cos^2 \frac{A}{2} - 1 = 1 - 2\sin^2 \frac{A}{2}$ .... (iii)

Also, $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} \qquad \text{..... (iv)}$$

[Where $A \neq n\pi + \frac{\pi}{2}$, ($n \in I$) and $A \neq (2n + 1)\pi$]

Again, $\sin A = \dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}}{1} = \dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}}$

[dividing numerator and denomenator by $\cos^2\frac{A}{2}$)

$$\sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

[where $A \neq (2n + 1)\pi$, $n \in I$]

Similarly, $\cos A = \dfrac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{1} = \dfrac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}}$

Now dividing numerator and denominator by $\cos^2\frac{A}{2}$

$\Rightarrow \quad \cos A = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$ [where $A \neq (2n + 1)\pi$, $n \in I$].

Example –1 : Find the values of

(i) $\cos 22\frac{1}{2}^{o}$ **(ii)** $\sin 15^{o}$

Solution : **(i)** We have c $\cos\frac{A}{2} = \sqrt{\dfrac{1+\cos A}{2}}$, putting $A = 45^{o}$

$$\cos 22\frac{1}{2}^{o} = \sqrt{\frac{1+\cos 45^{o}}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

(ii) $\sin 15^{o} = \sin(45^{o} - 30^{o})$

$= \sin 45^{o}.\cos 30^{o} - \cos 45^{o}.\sin 30^{o}$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Example – 2: Prove that sinA.sin (60º – A).sin(60º + A) = $\frac{1}{4}$ sin3A

Solution : sinA.sin (60º – A) sin (60º + A)

$$= \sin A.\,(\sin^2 60^\circ - \sin^2 A) \qquad [\because \sin(A+B).\sin(A-B) = \sin^2 A - \sin^2 B]$$

$$= \sin A\left[\left(\frac{\sqrt{3}}{2}\right)^2 - \sin^2 A\right] = \sin\left[\frac{3}{4} - \sin^2 A\right] = \frac{1}{4} \qquad [3\sin A - 4\sin^3 A]$$

$$= \frac{1}{4}\sin 3A$$

Example – 3: Prove that sin20º.sin40º.sin60º.sin80º = $\frac{3}{16}$

Solution : sin 60º. sin 20º.sin40º. sin 80º

$$= \frac{\sqrt{3}}{2}[\sin A.\,\sin(60^\circ - A).\,\sin(60^\circ + A)] \text{ where } A = 20^\circ$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{4}\cdot\sin 3A = \frac{\sqrt{3}}{8}\cdot\sin 60 = \frac{\sqrt{3}}{8}\cdot\frac{\sqrt{3}}{2} = \frac{3}{16}$$

Example – 4: If A + B + C = π and cos A = cos B . cos C show that tan B + tan C = tan A

Solution : L.H.S. = tan B + tan C

$$= \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C} = \frac{\sin B\,.\cos C + \cos B\,.\sin C}{\cos B\,.\cos C}$$

$$= \frac{\sin(B+C)}{\cos B\,.\cos C} = \frac{\sin(\pi - A)}{\cos B\,.\cos C} = \frac{\sin A}{\cos A} = \tan A = \text{R.H.S. (Proved)}$$

Examples – 5: Prove the followings

(a) $\cot 7\frac{1}{2}^\circ = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2$

(b) $\tan 37\frac{1}{2}^\circ = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2$

Solution : **(a)** We know $\cot\frac{\theta}{2} = \frac{1 + \cos\theta}{\sin\theta}$ (Choosing θ = 15)

$$= \cot 7\frac{1}{2}^\circ = \frac{1 + \cos 15^\circ}{\sin 15^\circ}$$

$$= \frac{1 + \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1}$$

$$= \frac{(2\sqrt{2} + \sqrt{3} + 1)(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{2\sqrt{6} + 2\sqrt{2} + \sqrt{3} + \sqrt{3} + 1 + 3}{3 - 1}$$

$$= \frac{2\sqrt{6} + 2\sqrt{3} + 2\sqrt{2} + 4}{2} = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2$$

(b) We know $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$ (Choosing $\theta = 75°$)

$$\tan 37\frac{1°}{2} = \frac{1-\cos 75°}{\sin 75°} = \frac{1-\cos(90°-15°)}{\sin(90°-15°)}$$

$$= \frac{1-\sin 15°}{\cos 15°} = \frac{1-\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{2\sqrt{2}-\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{(2\sqrt{2}-\sqrt{3}+1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \sqrt{6}+\sqrt{3}-\sqrt{2}-2$$

Example – 6: If sin A = K sin B, prove that $\tan\frac{1}{2}(A-B) = \frac{K-1}{K+1}\tan\frac{1}{2}(A+B)$

Solution : Given sin A = K sin B

$$\Rightarrow \frac{\sin A}{\sin B} = \frac{K}{1}$$

Using componendo & dividendo

$$\frac{\sin A+\sin B}{\sin A-\sin B} = \frac{K+1}{K-1}$$

$$\Rightarrow \frac{2\sin\frac{A+B}{2}.\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}.\sin\frac{A-B}{2}} = \frac{K+1}{K-1}$$

$$\Rightarrow \tan\frac{A+B}{2}.\cot\frac{A-B}{2} = \frac{K+1}{K-1}$$

$$\Rightarrow \tan\frac{A+B}{2} = \frac{K+1}{K-1}.\tan\frac{A-B}{2}$$

$$\Rightarrow \tan\frac{A-B}{2} = \frac{K-1}{K+1}\tan\frac{A+B}{2}$$

$\therefore$ L.H.S. = R.H.S. (Proved)

Example – 7: If $(1-e)\tan^2\frac{\beta}{2} = (1+e)\tan^2\frac{\alpha}{2}$, Prove that $\cos\beta = \frac{\cos\alpha - e}{1-e\cos\alpha}$

Solution : $(1-e)\tan^2\frac{\beta}{2} = (1+e)\tan^2\frac{\alpha}{2}$ (Given)

$$\tan^2\frac{\beta}{2} = \frac{1+e}{1-e}\tan^2\frac{\alpha}{2}$$

L.H.S = cos β

$$= \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}} = \frac{1-\frac{1+e}{1-e}\tan^2\frac{\alpha}{2}}{1+\frac{1+e}{1-e}\tan^2\frac{\alpha}{2}}$$

$$= \frac{1 - e - \tan^2\frac{\alpha}{2} - e\tan^2\frac{\alpha}{2}}{1 - e + \tan^2\frac{\alpha}{2} + e\tan^2\frac{\alpha}{2}} = \frac{\left(1 - \tan^2\frac{\alpha}{2}\right) - e\left(1 + \tan^2\frac{\alpha}{2}\right)}{\left(1 + \tan^2\frac{\alpha}{2}\right) - e\left(1 - \tan^2\frac{\alpha}{2}\right)}$$

$$= \frac{\dfrac{1 - \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}} - e\dfrac{1 + \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}}}{\dfrac{1 + \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}} - e\dfrac{1 - \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}}}$$

$$= \frac{\cos\alpha - e}{1 - e\cos\alpha} = \text{R.H.S (Proved)}$$

Example – 8: If A + B + C = π, then Prove the following

(i) sin 2A + sin 2B + sin 2C = 4 sin A. sin B. sin C

Solution : L.H.S. = sin 2A + sin 2B + sin 2C

= 2 sin (A + B) . cos (A – B) + 2 sin C. cos C

= 2 sin (π – C) . cos (A – B) + 2 sin C . cos C

= 2 sin C . cos (A – B) + 2 sin C. cos C

= 2 sin C [cos (A – B) + cos C]

= 2 sin C [cos (A – B) – cos (A + B)]

= 2 sin C . 2 sin A . sin B

= 4 sin A . sin B . sin C R.H.S. (Proved)

(ii) sin 2A + sin 2B – sin 2C = 4 cos A . cos B. sin C

Solution : L.H.S. = sin 2A + sin 2B – sin 2C

= 2 sin (A + B) . cos (A – B) – 2 sin C. cos C

= 2 sin (π – C) . cos (A – B) – 2 sin C. cos C

= 2 sin C. cos (A – B) – 2 sin C. cos C

= 2 sin C [cos (A – B) – cos {π – (A + B)}]

= 2 sin C {cos (A – B) + cos (A + B)}

$$= 2\sin C\left\{2\cos\frac{A - B + A + B}{2} . \cos\frac{A - B - A - B}{2}\right\}$$

= 4 sin C. cos A. cos B.

(iii) sin A + sin B – sin C = 4 sin $\frac{A}{2}$ sin $\frac{B}{2}$ cos $\frac{C}{2}$

Solution : L.H.S. = sin A + sin B – sin C

$$= 2\sin\frac{A + B}{2} . \cos\frac{A - B}{2} - 2\sin\frac{C}{2} . \cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2} . \cos\frac{A - B}{2} - 2\sin\frac{C}{2} . \cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} - \sin\frac{C}{2}\right\}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} - \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right\}$$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right\}$$

$$= 2\cos\frac{C}{2}\left\{(-2)\sin\left(\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\right) . \sin\left(\frac{\frac{A-B}{2}-\frac{A+B}{2}}{2}\right)\right\}$$

$$= -4\cos\frac{C}{2} . \sin\frac{A}{2} . \sin\left(-\frac{B}{2}\right)$$

$$= 4\cos\frac{C}{2} . \sin\frac{A}{2} . \sin\frac{B}{2} = \text{R.H.S (Proved)}$$

ASSIGNMENT

1. If $\tan\alpha = \frac{1}{2}$, $\tan\beta = \frac{1}{3}$, then find the value of $(\alpha + \beta)$

2. Find the value of $\dfrac{\cos 15^\circ + \sin 15^\circ}{\cos 15^\circ - \sin 15^\circ}$

3. Prove that $\dfrac{1}{\tan 3A - \tan A} - \dfrac{1}{\cot 3A - \cot A} = \cot 2A$

4. If A + B = 45º, show that (1 + tanA) (1+ tanB) = 2

5. If $(1 - e)\tan^2\frac{\beta}{2} = (1 + e)\tan^2\frac{\alpha}{2}$

   Prove that $\cos\beta = \dfrac{\cos\alpha - e}{1 - e\cos\alpha}$

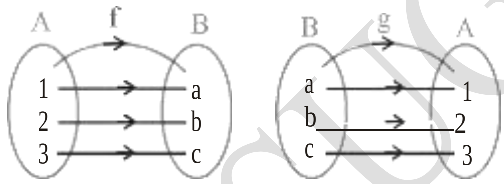
6. If $A + B + C = \pi$, prove that
   cos2A + cos2B + cos 2C + 1 + 4 cos A.cosB.cosC = 0

CHAPTER – 6

INVERSE TRIGONOMETRIC FUNCTIONS

INVERSE FUNCTION :

If $f : A \rightarrow B$ be a bijective function or one to one onto function from set A to the set B. As the function is 1 – 1, every element of A is associated with a unique element of B. As the function is onto, there is no element of B which in not associated with any element of A. Now if we consider a function g from B to A, we have for $f \in B$ there is unique $x \in A$. This g is called inverse function of f and is denoted by f^{-1}.



INVERSE TRIGONOMETRIC FUNCTION :

We know the equation $x = \sin y$ means that y is the angle whose sine value is x then we have $y = \sin^{-1}x$ similarly $y = \cos^{-1}x$ if $x = \cos y$ and $y = \tan^{-1}x$ is $x = \tan y$ etc.

The function $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$, $\mathrm{cosec}^{-1}x$, $\cot^{-1}x$ are called inverse trigonometric function.

* Properties of inverse trigonometric function.

I. Self adjusting property :

(i) $\sin^{-1}(\sin\theta) = \theta$

(ii) $\cos^{-1}(\cos\theta) = \theta$

(iii) $\tan^{-1}(\tan\theta) = \theta$

Proof:

(i) Let $\sin\theta = x$, then $\theta = \sin^{-1}x$

$\therefore \sin^{-1}(\sin\theta) = \sin^{-1}x = \theta$

proofs of (ii) * (iii) as above.

II. Reciprocal Property :

(i) $\mathrm{cosec}^{-1}\frac{1}{x} = \sin^{-1}x$

(ii) $\sec^{-1}\frac{1}{x} = \cos^{-1}x$

(iii) $\cot^{-1}\frac{1}{x} = \tan^{-1}x$

Proof :

(i) Let $x = \sin\theta$ then $\operatorname{cosec}\theta = \frac{1}{x}$

so that $\theta = \sin^{-1}x$ & $\theta = \operatorname{cosec}^{-1}\frac{1}{x}$

$\therefore \sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}$

(ii) and (iii) may be proved similarly

III. Conversion property :

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

(ii) $\cos^{-1}x\ \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$

Proof:

(i) Let $\theta = \sin^{-1}x$ so that $\sin\theta = x$

Now $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$

i.e., $\theta = \cos^{-1}\sqrt{1-x^2}$

Also $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \theta = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

Thus we have $\theta = \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

Theorem – 1 : Prove that

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(iii) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$

Proof :

(i) Let $\sin^{-1}x = \theta$, then

$x = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow \cos^{-1}x = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sin^{-1}x$

$\Rightarrow \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

(ii) and (iii) can be proved similarly.

Theorem – 2 : If $xy < 1$, then

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Proof : Let $\tan^{-1}x = \theta_1$ and $\tan^{-1}y = \theta_2$

Then

$\tan\theta_1 = x$ and $\tan\theta_2 = y$

$$\Rightarrow \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{x+y}{1-xy}$$

$$\Rightarrow \theta_1 + \theta_2 = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Theorem – 3 : $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Proof : Let $\tan^{-1}x = \theta_1$ and $\tan^{-1}y = \theta_2$

$\Rightarrow \tan\theta_1 = x$ and $\tan\theta_2 = y$

$$\Rightarrow \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \frac{x-y}{1+xy}$$

$$\Rightarrow \theta_1 - \theta_2 = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$\Rightarrow \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

Note : $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

Theorem – 4 : Prove that :

(i) $2\sin^{-1}x = \sin^{-1}\left[2x\sqrt{1-x^2}\right]$

(ii) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$

Proof :

(i) Let $\sin^{-1}x = \theta$, Then, $x = \sin\theta$

$\therefore \sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta.\sqrt{1-\sin^2\theta}$

$= 2x\sqrt{1-x^2}$

$\Rightarrow 2\theta = \sin^{-1}\left[2x\sqrt{1-x^2}\right] \Rightarrow 2\sin^{-1}x = \sin^{-1}\left[2x\sqrt{1-x^2}\right]$

(ii) Let $\cos^{-1}x = \theta$ Then, $x = \cos\theta$

$\therefore \cos 2\theta = (2\cos^2\theta - 1) = 2x^2 - 1$

$\Rightarrow 2\theta = \cos^{-1}(2x^2 - 1)$

$\Rightarrow 2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$

Theorem – 5 : Prove that

(i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{(1-x^2)(1-y^2)}\right]$

(iii) $\sin^{-1}x - \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{(1-x^2)(1-y^2)}\right]$

Proof :

(i) Let $\sin^{-1}x = \theta_1$, and $\sin^{-1}y = \theta_2$, Then

$\sin\theta_1 = x$ and $\sin\theta_2 = y$

$\therefore \sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$

$= \sin\theta_1\sqrt{1-\sin^2\theta_2} + \sqrt{(1-\sin^2\theta_1)}\ \sin\theta_2$

$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \theta_1 + \theta_2 = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$

$\Rightarrow \sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$

The other results may be proved similarly.

Example – 1 : **If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ then prove that $x^2 + y^2 + z^2 + 2xyz = 1$**

Solution : Given $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$\cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = (\pi - \cos^{-1}z)$

$xy - \sqrt{1-x^2}\sqrt{1-y^2} = \cos(\pi - \cos^{-1}z)$

$\Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = -\cos(\cos^{-1}z) = -z$

$\Rightarrow xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$

$\Rightarrow (xy + z)^2 = (1-x^2)(1-y^2) = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$ (Proved)

Example – 2 : **Find the value of $\cos\tan^{-1}\cot\cos^{-1}\frac{\sqrt{3}}{2}$**

Solution : $\cos^{-1}\frac{\sqrt{3}}{2} = \theta \Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \theta = \frac{\pi}{6} \Rightarrow \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

$\therefore \cos\tan^{-1}\cot\cos^{-1}\frac{\sqrt{3}}{2} = \cos\tan^{-1}\cot\frac{\pi}{6}$

$= \cos\tan^{-1}\sqrt{3}\ \left[\because \tan^{-1}\sqrt{3} = \frac{\pi}{3}\right] = \cos\frac{\pi}{3} = \frac{1}{2}$

Example – 3 : Prove that $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$.

Solution : L.H.S $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2}\right]$$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{\frac{1}{2}+\frac{1}{7}}{1-\frac{1}{2}\times\frac{1}{7}}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{\frac{9}{14}}{\frac{13}{14}} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{9}{13}$$

$$= \tan^{-1}\frac{\frac{1}{2}+\frac{9}{13}}{1-\frac{1}{2}\times\frac{9}{13}} = \tan^{-1}\frac{\frac{31}{26}}{\frac{17}{26}} = \tan^{-1}\frac{31}{17} = \text{R.H.S. (Proved)}$$

Example – 4 : Prove that $\cot^{-1}9 + \operatorname{cosec}^{-1}\frac{\sqrt{41}}{4} = \frac{\pi}{4}$

Solution : L.H.S. $= \cot^{-1}9 + \operatorname{cosec}^{-1}\frac{\sqrt{41}}{4}$

$$= \tan^{-1}\frac{1}{9} + \tan^{-1}\frac{4}{5} \qquad \left[\because \operatorname{cosec}^{-1}\frac{\sqrt{41}}{4} = \tan^{-1}\frac{4}{5}\right]$$

$$= \tan^{-1}\frac{\frac{1}{9}+\frac{4}{5}}{1-\frac{1}{9}\cdot\frac{4}{5}} = \tan^{-1}\frac{\frac{5+36}{45}}{\frac{45-4}{45}} = \tan^{-1}\frac{\frac{41}{45}}{\frac{41}{45}}$$

$$= \tan^{-1}1 = \frac{\pi}{4} \text{ R.H.S. (Proved)}$$

ASSIGNMENT

1. Find the value of $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3$

2. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$. Show that

$$x.\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

3. If s $\sin^{-1}\frac{x}{5} + \operatorname{cos\,ec}^{-1}\frac{5}{4} = \frac{\pi}{2}$. Find the value of x.