LECTURE NOTE ON NUMERICAL ANALYSIS

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FINITE DIFFERENCES & INTERPOLATION

The study of finite differences deals with the changes that take place in the function due to the finite change in independent variable.

Let $y = f(x)$ be the function.

Let $x: x_0 \quad x_0+h, \quad x_0+2h, \ldots \quad x_0+nh$ (Arguments)

$y: y_0 \quad y_1, \quad y_2 \quad \ldots \quad y_n$ (Entries)

h is the number by which the values of the argument advance is called the interval of differencing.

TYPES OF DIFFERENCES

1) Forward differences (Δ)
2) Backward differences (∇)
3) Shift operator (E)

FORWARD DIFFERENCES

1st order $\rightarrow$ $\boxed{\Delta(f(x)) = f(x+h) - f(x)}$

Ex:- (1) $\Delta(x) = x+h-x$

$= h$

(2) $\Delta(\cos x) = \cos(x+h) - \cos x$

$= 2\sin\frac{x+h+x}{2} \cdot \sin\frac{x-x-h}{2}$

$= 2\sin\frac{2x+h}{2} \cdot \sin\left(-\frac{h}{2}\right)$

$= -2\sin\frac{2x+h}{2} \cdot \sin\frac{h}{2}$

(3) $\Delta(\tan^{-1}x) = \tan^{-1}(x+h) - \tan^{-1}x$

$= \tan^{-1}\left(\frac{x+h-x}{1+(x+h)x}\right)$

$= \tan^{-1}\left(\frac{h}{1+x^2+xh}\right)$

$\Delta y_0 = y_1 - y_0$

$\Delta y_1 = y_2 - y_1$

$\Delta y_2 = y_3 - y_2$

$\vdots$

$\Delta y_{n-1} = y_n - y_{n-1}$

2nd order $\boxed{\Delta^2(f(x)) = \Delta(f(x+h)) - \Delta(f(x))}$

Ex-1 $\Delta^2(\cos 2x) = \Delta(\cos 2(x+h)) - \Delta(\cos 2x)$

$= \Delta(\cos(2x+2h)) - \Delta(\cos 2x)$

$= [\cos(2(x+h)+2h) - \cos(2x+2h)]$

$- (\cos 2(x+h) - \cos 2x)$

$= \cos(2x+2h+2h) - \cos(2x+2h) - \cos(2x+2h) + \cos 2x$

$= \cos(2x+4h) - 2\cos(2x+2h) + \cos 2x$

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$

$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$

$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$

$\vdots$

$\Delta^2 y_{n-2} = \Delta y_{n-1} - \Delta y_{n-2}$

Similarly we can find different order of Forward differences.

Forward differences Table

x	y	Δ	Δ^2	Δ^3
x_0	y_0	$\Delta y_0 = y_1 - y_0$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_0+h	y_1	$\Delta y_1 = y_2 - y_1$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
x_0+2h	y_2	$\Delta y_2 = y_3 - y_2$		
x_0+3h	y_3			

y_0 is called the leading term

$\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 \cdots$ are called the leading difference.

BACKWARD DIFFERENCES

1st order $\boxed{\nabla(f(x)) = f(x) - f(x-h)}$

$$\nabla y_1 = y_1 - y_0$$
$$\nabla y_2 = y_2 - y_1$$
$$\nabla y_3 = y_3 - y_2$$
$$\vdots$$
$$\nabla y_n = y_n - y_{n-1}$$

2nd order $\boxed{\nabla^2(f(x)) = \nabla(f(x)) - \nabla(f(x-h))}$

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$
$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$$
$$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$$
$$\vdots$$
$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

Similarly we can find different order of Backward differences.

BACKWARD DIFFERENCES TABLE

x	y	∇	∇^2	∇^3
x_0	y_0			
		$\nabla y_1 = y_1 - y_0$		
x_0+h	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
		$\nabla y_2 = y_2 - y_1$		
x_0+2h	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		
x_0+3h	y_3			

DIFFERENCES OF A POLYNOMIAL

If $f(x)$ is a polynomial of degree 'n'

Then 1st order difference is a polynomial of degree $n-1$

2nd order difference is a polynomial of degree $n-2$.

⋮

nth order difference is a constant

$(n+1)$th order difference is zero.

Note → if 'n' no. of entries are given $(y_0, y_1, y_2 \cdots y_n)$ then we can form a $(n-1)$ degree polynomial. so its nth order differences are zero.

SHIFT OPERATOR (E)

The shift operator (E) is defined as that operator which when applied to a function, increases the value of the argument (x) by one interval (h).

$$E\, f(x) = f(x+h) \qquad E y_0 = y_1$$
$$E\, f(x+h) = f(x+2h) \qquad E y_1 = y_2$$
$$E\, f(x+2h) = f(x+3h) \qquad E y_2 = y_3$$

$$E^2(f(x)) = E\,(E\, f(x)) \qquad E^2 y_0 = y_2$$
$$= E(f(x+h)) \qquad E^2 y_1 = y_3$$
$$= f(x+2h)$$

$$E^2(f(x+h)) = f(x+3h)$$

Similarly

$$E^{-1}(f(x)) = f(x-h) \qquad E^{-1} y_1 = y_0$$
$$E^{-1}(f(x+h)) = f(x) \qquad E^{-1} y_2 = y_1$$

Here E^{-1} is called as inverse shift operator.

Relation between the operators

(i) $E = 1 + \Delta$ or $\Delta = E - 1$

(ii) $E^{-1} = 1 - \nabla$ or $\nabla = 1 - E^{-1}$

(iii) $\Delta = E\nabla = \nabla E$

Proof (i) We know that

$$\Delta f(x) = f(x+h) - f(x)$$
$$= E f(x) - f(x)$$
$$= (E-1) f(x)$$

$$\Rightarrow \Delta = E - 1 \text{ or } E = 1 + \Delta$$

(ii) We know that

$$\nabla f(x) = f(x) - f(x-h)$$
$$= f(x) - E^{-1} f(x)$$
$$= (1 - E^{-1}) f(x)$$

$$\Rightarrow \nabla = 1 - E^{-1} \text{ or } E^{-1} = 1 - \nabla$$

(iii) We know that $\nabla f(x) = f(x) - f(x-h)$

$$\Rightarrow E\nabla f(x) = E f(x) - E f(x-h)$$
$$= f(x+h) - f(x)$$
$$= \Delta f(x)$$

$$\therefore E\nabla = \Delta \quad \text{——} \quad (1)$$

again $E f(x) = f(x+h)$

$$\Rightarrow \nabla E f(x) = \nabla f(x+h)$$
$$= f(x+h) - f(x)$$
$$= \Delta f(x)$$

$$\therefore \nabla E = \Delta \quad \text{——} \quad (2)$$

From (1) and (2) $\Delta = E\nabla = \nabla E$

Ex-1 Construct the forward differences for the data below:

x: 0 1 2 3 4

$f(x)$: 1 1.5 2.2 3.1 4.6

Evaluate $\Delta^3 f(1)$

Sol^n

x	y	Δ	Δ^2	Δ^3	Δ^4
0	1				
1	1.5	0.5	0.2		
2	2.2	0.7	0.2	0	0.4
3	3.1	0.9	0.6	0.4	
4	4.6	1.5			

$\boxed{\Delta^3 y_1 = \Delta^3 f(1)}$

Here $f(0) = 1$
$f(1) = 1.5$
$f(2) = 2.2$
$f(3) = 3.1$
$f(4) = 4.6$

$$
\begin{aligned}
\Delta^3 f(1) &= \Delta^2 \cdot (\Delta f(1)) \\
&= \Delta^2 \cdot (f(2) - f(1)) \\
&= \Delta^2 f(2) - \Delta^2 f(1) \\
&= \Delta(\Delta f(2)) - \Delta(\Delta f(1)) \\
&= \Delta(f(3) - f(2)) - \Delta(f(2) - f(1)) \\
&= \Delta f(3) - \Delta f(2) - \Delta f(2) + \Delta f(1) \\
&= \Delta f(3) - 2\Delta f(2) + \Delta f(1) \\
&= (f(4) - f(3)) - 2(f(3) - f(2)) + (f(2) - f(1)) \\
&= f(4) - f(3) - 2f(3) + 2f(2) + f(2) - f(1) \\
&= f(4) - 3f(3) + 3f(2) - f(1) \\
&= 4.6 - 3 \times 3.1 + 3 \times 2.2 - 1.5 \\
&= 4.6 - 9.3 + 6.6 - 1.5 \\
&= 11.2 - 10.8 \\
&= 0.4
\end{aligned}
$$

Ex-2 if $u_0 = 3$ $u_1 = 12$ $u_2 = 81$ $u_3 = 2000$, $u_4 = 100$
Calculate $\Delta^4 u_0$

Sol^n $\Delta^4 u_0 = \Delta^3 u_1 - \Delta^3 u_0$

$= (\Delta^2 u_2 - \Delta^2 u_1) - (\Delta^2 u_1 - \Delta^2 u_0)$

$= \Delta^2 u_2 - 2\Delta^2 u_1 + \Delta^2 u_0$

$= (\Delta u_3 - \Delta u_2) - 2(\Delta u_2 - \Delta u_1) + (\Delta u_1 - \Delta u_0)$

$= \Delta u_3 - \Delta u_2 - 2\Delta u_2 + 2\Delta u_1 + \Delta u_1 - \Delta u_0$

$= \Delta u_3 - 3\Delta u_2 + 3\Delta u_1 - \Delta u_0$

$= (u_4 - u_3) - 3(u_3 - u_2) + 3(u_2 - u_1) - (u_1 - u_0)$

$= u_4 - u_3 - 3u_3 + 3u_2 + 3u_2 - 3u_1 - u_1 + u_0$

$= u_4 - 4u_3 + 6u_2 - 4u_1 + u_0$

$= 100 - 4 \times 2000 + 6 \times 81 - 4 \times 12 + 3$

$= 100 - 8000 + 486 - 48 + 3$

$= -7459.$

Ex:-3 show that $\Delta^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$

Sol^n $\Delta^3 y_i = \Delta^2 y_{i+1} - \Delta^2 y_i$

$= (\Delta y_{i+2} - \Delta y_{i+1}) - (\Delta y_{i+1} - \Delta y_i)$

$= \Delta y_{i+2} - 2\Delta y_{i+1} + \Delta y_i$

$= (y_{i+3} - y_{i+2}) - 2(y_{i+2} - y_{i+1}) + (y_{i+1} - y_i)$

$= y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$ (proved)

Ex-4. Form the table of backward differences of the function. $f(x) = x^3 - 3x^2 - 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$

Soln $y = f(x) = x^3 - 3x^2 - 5x - 7$

$y_{-1} = f(-1) = (-1)^3 - 3(-1)^2 - 5(-1) - 7 = -6$

$y_0 = f(0) = 0 - 0 - 0 - 7 = -7$

$y_1 = f(1) = -14$

$y_2 = f(2) = -21$

$y_3 = f(3) = -22$

$y_4 = f(4) = -11$

$y_5 = f(5) = 18$

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
-1	-6						
		-1					
0	-7		-6				
		-7		6			
1	-14		0		0		
		-7		6		0	
2	-21		6		0		0
		-1		6		0	
3	-22		12		0		
		11		6			
4	-11		18				
		29					
5	18						

Ex-5 form a table of differences for the function $f(x) = x^3 + 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$. Continue the table to obtain $f(6)$

Solution :- $f(x) = x^3 + 5x - 7$

$y_{-1} = f(-1) = -13$

$y_0 = f(0) = -7$

$y_1 = f(1) = -1$

$y_2 = f(2) = 11$

$y_3 = f(3) = 35$

$y_4 = f(4) = 77$

$y_5 = f(5) = 143$

x	y	Δ	Δ^2	Δ^3	Δ^4
-1	-13				
		6			
0	-7		0		
		6		6	
1	-1		6		0
		12		6	
2	11		12		0
		24		6	
3	35		18		0
		42		6	
4	77		24		$y_6 - 239$
		66		$y_6 - 233$	
5	143		$y_6 - 209$		
		$y_6 - 143$			
6	y_6				

Since $f(x)$ is a polynomial of degree '3' it's 4th order differences are zero.

Hence $y_6 - 239 = 0$

$\Rightarrow y_6 = 239$

$\Rightarrow f(6) = 239$ (Ans)

Imp

Ex-6 By Forming a difference table find the missing values in the following table.

x:	0	1	2	3	4
y:	1 (y_0)	3 (y_1)	9 (y_2)	?	81 (y_4)

Let the missing value is y_3

x	y	Δ	Δ^2	Δ^3	Δ^4
0	1				
		2			
1	3		4		
		6		$y_3 - 19$	
2	9		$y_3 - 15$		$124 - 4y_3$
		$y_3 - 9$		$105 - 3y_3$	
3	y_3		$90 - 2y_3$		
		$81 - y_3$			
4	81				

Here four entries are given (y_0, y_1, y_2, y_4) so y can be represented by a 3rd degree polynomial hence 4th order difference becomes zero.

$\Rightarrow 124 - 4y_3 = 0$

$\Rightarrow y_3 = 31$

$\therefore$ Hence the missing term is 31

Ex-7 By Forming a difference table find find the missing values in the following table assuming that the Fourth differences are equal to zero.

x:	0	5	10	15	20	25
y:	6	10	—	17	—	31
	y_0	y_1	y_2	y_3	y_4	y_5

Sol^n Let the missing values are y_2 and y_4

x	y	Δ	Δ^2	Δ^3	Δ^4
0	6				
		4			
5	10		y_2-14	$41-3y_2$	
		y_2-10			y_4+6y_2-102
10	y_2		$27-2y_2$	y_4+3y_2-61	
		$17-y_2$			
15	17		y_4+y_2-34		$143-4y_4-4y_2$
		y_4-17		$82-3y_4-y_2$	
20	y_4		$48-y_4-y_4$		
		$31-y_4$			
25	31				

Given Fourth differences are equal to zero

$$\Rightarrow \Delta^4 = 0$$

$$\Rightarrow y_4 + 6y_2 - 102 = 0 \quad \text{——} (1)$$

and $143 - 4y_4 - 4y_2 = 0 \quad$ —— (2)

Solving (1) and (2)

$$eq^n (1) \times 4 \Rightarrow \cancel{4y_4} + 24\,y_2 - 408 = 0$$

$$eq^n (2) \times 1 \Rightarrow -\cancel{4y_4} - 4y_2 + 143 = 0$$

$$20y_2 - 265 = 0$$

$$20y_2 = 265$$

$$y_2 = \frac{265}{20}$$

$$\boxed{y_2 = 13.25}$$

Putting y_2 in eq^n (1)

$$\Rightarrow y_4 + 6 \times 13.25 - 102 = 0$$

$$\Rightarrow y_4 + 79.50 - 102 = 0$$

$$\Rightarrow y_4 - 22.50 = 0 \quad \Rightarrow \boxed{y_4 = 22.50}$$

Ex-8 Find the missing values in the following table

x:	0	1	2	3	4	5	6
y:	5	11	22	40	—	140	—
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Sol^n Let the missing values are y_4 and y_6

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	5	6	5	2	y_4-67	$370-5y_4$
1	11	11	7	y_4-65	$303-4y_4$	y_6+10y_4-1001
2	22	18	y_4-58	$238-3y_4$	y_6+6y_4-698	
3	40	y_4-40	$180-2y_4$	y_6+3y_4-460		
4	y_4	$140-y_4$	y_6+y_4-280			
5	140	y_6-140				
6	y_6					

Here Five entries y_0, y_1, y_2, y_3, y_5 are given the function y can be represented by 4th degree polynomial and ~~hece~~ hence 5th difference becomes zero.

$$\Delta^5 = 0$$

$$\Rightarrow 370 - 5y_4 = 0 \quad \text{——— (1)}$$

$$\text{and} \quad y_6 + 10y_4 - 1001 = 0 \quad \text{——— (2)}$$

From eq^n (1) $370 = 5y_4$

$$y_4 = \frac{370}{5}$$

$$\boxed{y_4 = 74}$$

Putting $y_4 = 74$ in eq^n (2)

$$\Rightarrow y_6 + 10 \times 74 - 1001 = 0$$

$$\Rightarrow y_6 + 740 - 1001 = 0$$

$$\Rightarrow y_6 - 261 = 0$$

$$\Rightarrow \boxed{y_6 = 261}$$

INTERPOLATION

Let for the function $y = f(x)$, the values of y are given for equidistant values of x as follows

x: x_0 x_1 x_2 ...... x_n

y: y_0 y_1 y_2 ...... y_n

The process of finding the value of y corresponding to any value of x between x_0 and x_n is called Interpolation.

1. Newton's Forward interpolation formula for equal intervals

$$\boxed{f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots\dots}$$

Where $x = x_0 + ph$

$\Rightarrow \boxed{p = \frac{x - x_0}{h}}$

Note $\rightarrow$ $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$

Example:- ① using Newton's Forward formula Find the value of $f(1.6)$ if

x: 1 1.4 1.8 2.2

y: 3.49 4.82 5.96 6.5

Sol^n

Here $\boxed{h = 0.4}$

x	y	Δ	Δ^2	Δ^3
x_0 1	y_0 3.49	Δy_0 1.33	$\Delta^2 y_0$ −0.19	$\Delta^3 y_0$ −0.41
1.4	4.82	1.14	−0.6	
1.8	5.96	0.54		
2.2	6.5			

$h = 1.4 - 1$
$h = 0.4$

Here $x = 1.6$ $x_0 = 1$, $y_0 = 3.49$ $\Delta y_0 = 1.33$ $\Delta^2 y_0 = -0.19$ $\Delta^3 y_0 = -0.41$

$p = \frac{x - x_0}{h}$

$= \frac{1.6 - 1}{0.4}$

$= 1.5$

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$f(1.6) = 3.49 + 1.5 \times 1.33 + \frac{(1.5)(1.5-1)}{2}.(-0.19) + \frac{(1.5)(1.5-1)(1.5-2)}{6} \times (-0.41)$$

$$= 3.419 + 1.9950 + (-0.0713) + 0.0256$$

$$= \boxed{5.4394} \text{ (Ans)}$$

Ex-2 using Newton's Forward interpolation formula from the following table find y at $x = 8$.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0 (x_0)	7 (y_0)	4 (Δy_0)	−1 ($\Delta^2 y_0$)	2 ($\Delta^3 y_0$)	−1 ($\Delta^4 y_0$)	0 ($\Delta^5 y_0$)
5	11	3	1	1	−1	
10	14	4	2	0		
15	18	6	2			
20	24	8				
25	32					

Common $h \rightarrow$ difference between values of x

$h = 5 - 0$

$h = 5$

Here $x = 8$, $h = 5$, $x_0 = 0$, $y_0 = 7$, $\Delta y_0 = 4$, $\Delta^2 y_0 = -1$, $\Delta^3 y_0 = 2$

$\Delta^4 y_0 = -1$, $\Delta^5 y_0 = 0$

$$p = \frac{x - x_0}{h} = \frac{8 - 0}{5} = 1.6$$

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$f(8) = 7 + 1.6 \times 4 + \frac{(1.6)(1.6-1)}{2} \times (-1) + \frac{(1.6)(1.6-1)(1.6-2)}{6} \times 2$$

$$+ \frac{(1.6)(1.6-1)(1.6-2)(1.6-3)}{24} \times -1$$

$$= 7 + 6.4 - 0.4800 - 0.1280 - 0.0224$$

$$= \boxed{12.7696}$$

Ex-3 Using Newton's Forward interpolation formula find y when $x=1.4$ if

x: 1.1 1.3 1.5 1.7 1.9

y: 0.21 0.69 1.25 1.89 2.61

Sol

x	y	Δ	Δ^2	Δ^3	Δ^4
1.1	0.21				
		0.48			
1.3	0.69		0.08		
		0.56		0	
1.5	1.25		0.08		0
		0.64		0	
1.7	1.89		0.08		
		0.72			
1.9	2.61				

$x_0 = 1.1$

Here $x=1.4$, $h=0.2$, $y_0=0.21$, $\Delta^2 y_0=0.08$

$$P = \frac{x-x_0}{h} = \frac{1.4-1.1}{0.2} = \frac{0.3}{0.2} = 1.5$$

$$f(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0$$

$$= 0.21 + 1.5 \times 0.48 + \frac{(1.5)(1.5-1)}{2} \times 0.08$$

$$= 0.21 + 0.72 + 0.03$$

$$= \boxed{0.96}$$

Ex-4) Using Newton's Forward interpolation formula find $f(1.28)$

if $f(1.15) = 1.0723$, $f(1.20) = 1.0954$

$f(1.25) = 1.1180$ & $f(1.30) = 1.1401$

Solution Here

x:	1.15	1.20	1.25	1.30
y:	1.0723	1.0954	1.1180	1.1401

x	y	Δ	Δ^2	Δ^3
1.15	1.0723			
		0.0231		
1.20	1.0954		-0.0005	
		0.0226		0
1.25	1.1180		-0.0005	
		0.0221		
1.30	1.1401			

Here $x = 1.28$ $h = 0.05$

$x_0 = 1.15$, $y_0 = 1.0723$; $\Delta y_0 = 0.0231$, $\Delta^2 y_0 = -0.0005$

$$P = \frac{x - x_0}{h}$$

$$= \frac{1.28 - 1.15}{0.05}$$

$$= 2.6$$

$$f(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0$$

$$= 1.0723 + 2.6 \times 0.0231 + \frac{(2.6)(2.6-1)}{2} \times (-0.0005)$$

$$= 1.0723 + 0.0601 \quad - 0.0010$$

$$= \boxed{1.1314}$$

Ex-5 Construct Newton's Forward Interpolation $\boxed{\text{Polynomial}}$ for the following data.

x: 4 6 8 10

y: 1 3 8 16

Hence evaluate $x = 5$

x	y	Δ	Δ^2	Δ^3
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

Here $x_0 = 4$, $y_0 = 1$ $\Delta y_0 = 2$ $\Delta^3 y_0 = 3$

$h = 2$

$$P = \frac{x - x_0}{h} = \frac{x-4}{2}$$

$$f(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0$$

$$= 1 + \frac{x-4}{2} \times 2 + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2} - 1\right)}{2} \times 3$$

$$= 1 + x - 4 + \frac{3}{2}\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)$$

$$= x - 3 + \frac{3}{2}\left(\frac{x^2 - 10x + 24}{4}\right)$$

$$= x - 3 + \frac{3}{8}(x^2 - 10x + 24)$$

$$= x - 3 + \frac{3}{8}x^2 - \frac{30}{8}x + \frac{72}{8}$$

$$= \frac{3}{8}x^2 + x - \frac{30}{8}x + 9 - 3$$

$$= \frac{3}{8}x^2 - \frac{22}{8}x + 6$$

$$= \boxed{\frac{3}{8}x^2 - \frac{11}{4}x + 6}$$

$\therefore$ at $x = 5$

$$f(5) = \frac{3}{8}(5)^2 - \frac{11}{4}(5) + 6$$

$$= 1.625 \text{ (Ans)}$$

NEWTON'S BACKWARD INTERPOLATION FORMULA

$$\boxed{f(x) = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n + \cdots}$$

Here $x = x_n + ph$

$$\boxed{P = \frac{x - x_n}{h}}$$

<u>Ex-1</u> Using Newtons Backward formula find the value of $f(105)$ if

x: 80 85 90 95 100

y: 5026 5674 6362 7088 7854

x	y	∇	∇^2	∇^3	∇^4
80	5026				
		648			
85	5674		40		
		688		-2	
90	6362		38		4 $\nabla^4 y_n$
		726		2 $\nabla^3 y_n$	
95	7088		40 $\nabla^2 y_n$		
		766 ∇y_n			
100 x_n	7854 y_n				

Here $x_n = 100$, $y_n = 7854$ $\nabla y_n = 766$ $\nabla^2 y_n = 40$, $\nabla^3 y_n = 2$

$\nabla^4 y_n = 4$

Given $x = 105$, $h = 5$

$$P = \frac{x - x_n}{h} = \frac{105 - 100}{5} = 1$$

$$f(x) = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n + \frac{P(P+1)(P+2)(P+3)}{4!}\nabla^4 y_n$$

$$f(105) = 7854 + 1\times 766 + \frac{1(2)}{2}\times 40 + \frac{1\times 2\times 3}{6}\times 2 + \frac{1\times 2\times 3\times 4}{24}\times 4$$

$$= 7854 + 766 + 40 + 2 + 4$$

$$= \boxed{8666} \text{ (Ans)}$$

Ex-2 Using Newton's Backward interpolation Formula find y when $x = 4.5$

if x : 1 2 3 4

y : 27 30 34 37

Sol^n

x	y	∇	∇^2	∇^3
1	27			
		3		
2	30		1	
		4		-2
3	34		-1	
		3		
4	37			

Here $p = \dfrac{x - x_n}{h} = \dfrac{4.5 - 4}{1} = \boxed{0.5}$

Here $x_n = 4$, $y_n = 37$, $\nabla y_n = 3$ $\nabla^2 y_n = -1$ $\nabla^3 y_n = -2$

Given $x = 4.5$ $h = 1$

$$f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$f(4.5) = 37 + 0.5 \times 3 + \frac{0.5(0.5+1)}{2} \times (-1) + \frac{0.5(0.5+1)(0.5+2)}{6} \times (-2)$$

$$= 37 + 1.5 - 0.3750 - 0.6250$$

$$= \boxed{37.5} \text{ (Ans)}$$

Ex-3 Find the cubic polynomial using Newton's Backward interpolation formula if

x : 0 1 2 3

y : 1 2 1 10

Hence evalute $f(4)$

Sol^n

x	y	∇	∇^2	∇^3
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

Here $x_n = 3$, $y_n = 10$, $\nabla y_n = 9$, $\nabla^2 y_n = 10$ $\nabla^3 y_n = 12$

$$P = \frac{x - x_n}{h} = \frac{x-3}{1} = x - 3$$

$$f(x) = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$$

$$= 10 + (x-3)\times 9 + \frac{(x-3)(x-3+1)}{2}\times 10 + \frac{(x-3)(x-3+1)(x-3+2)}{6}\times 12$$

$$= 10 + 9x - 27 + (x-3)(x-2)\times 5 + (x-3)(x-2)(x-1)\times 2$$

$$= 9x - 17 + 5(x^2 - 5x + 6) + 2(x-1)(x^2 - 5x + 6)$$

$$= 9x - 17 + 5x^2 - 25x + 30 + (2x-2)(x^2 - 5x + 6)$$

$$= 5x^2 - 16x + 13 + 2x^3 - 10x^2 + 12x - 2x^2 + 10x - 12$$

$=$

$$= \boxed{2x^3 - 7x^2 + 6x + 1}$$

$$f(4) = 2\times 4^3 - 7\times 4^2 + 6\times 4 + 1$$

$= 128 - 112 + 24 + 1$

$= 153 - 112$

$= \boxed{41}$ (Ans)

LAGRANGE'S INTERPOLATION FORMULA

Let $x: \; x_0 \; x_1 \; x_2 \; x_3 \; \cdots\cdots \; x_n$

$y: \; y_0 \; y_1 \; y_2 \; y_3 \; \cdots\cdots \; y_n$

Then

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\cdots\cdots(x_0-x_n)} \times y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots\cdots(x_1-x_n)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\cdots\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots\cdots(x_2-x_n)} \times y_2$$

$$\vdots$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)(x_n-x_3)\cdots(x_n-x_{n-1})} \times y_n$$

Example-1 Use Lagrange's interpolation formula to find the value of y when $x=10$ if

$x: 5 \quad 6 \quad 9 \quad 11$

$y: 12 \quad 13 \quad 14 \quad 16$

Sol^n Given $x_0=5$, $x_1=6$ $x_2=9$ $x_3=11$

$y_0=12$ $y_1=13$ $y_2=14$ $y_3=16$

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$= \frac{4 \times 1 \times -1}{-1 \times -4 \times -6} \times 12 + \frac{5 \times 1 \times -1}{1 \times -3 \times -5} \times 13$$

$$+ \frac{5 \times 4 \times -1}{4 \times 3 \times -2} \times 14 + \frac{5 \times 4 \times 1}{6 \times 5 \times 2} \times 16$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3}$$

$$= 2 + \frac{38}{3}$$

$$= \frac{44}{3}$$

$$= \boxed{14.66}$$

Ex-2 if $y(1) = -3$ $y(3) = 9$ $y(4) = 30$, $y(6) = 132$ Using Lagrange's Interpolation formula find $y(5)$

Sol^n Given $x_0 = 1$ $x_1 = 3$ $x_2 = 4$ $x_3 = 6$

$y_0 = -3$ $y_1 = 9$ $y_2 = 30$ $y_3 = 132$

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$f(5) = \frac{(5-3)(5-4)(5-6)}{(1-3)(1-4)(1-6)} \times -3 + \frac{(5-1)(5-4)(5-6)}{(3-1)(3-4)(3-6)} \times 9$$

$$+ \frac{(5-1)(5-3)(5-6)}{(4-1)(4-3)(4-6)} \times 30 + \frac{(5-1)(5-3)(5-4)}{(6-1)(6-3)(6-4)} \times 132$$

$$= \frac{2 \times 1 \times -1}{-2 \times -3 \times -5} \times -3 + \frac{4 \times 1 \times -1}{2 \times -1 \times -3} \times 9$$

$$+ \frac{4 \times 2 \times -1}{3 \times 1 \times -2} \times 30 + \frac{4 \times 2 \times 1}{5 \times 3 \times 2} \times 132$$

$$= -\frac{1}{5} - 6 + 40 + \frac{176}{5}$$

$$= -0.2 - 6 + 40 + 35.2$$

$$= \boxed{69} \text{ Ans.}$$

Ex:-3 Using Lagrange's interpolation formula find a polynomial if

$$x: \quad 0 \quad 2 \quad 3$$
$$y: \quad -4 \quad 2 \quad 14$$

Sol: Given $x_0 = 0 \quad x_1 = 2 \quad x_2 = 3$

$y_0 = -4 \quad y_1 = 2 \quad y_2 = 14$

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \times y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \times y_2$$

$$f(x) = \frac{(x-2)(x-3)}{(0-2)(0-3)} \times (-4) + \frac{(x-0)(x-3)}{(2-0)(2-3)} \times 2$$

$$+ \frac{(x-0)(x-2)}{(3-0)(3-2)} \times 14$$

$$= \frac{x^2-5x+6}{6} \times (-4) + \frac{x(x-3)}{2 \times -1} \times 2 + \frac{x(x-2)}{3 \times 1} \times 14$$

$$= \frac{-2}{3}(x^2-5x+6) - (x^2-3x) + \frac{14}{3}(x^2-2x)$$

$$= -\frac{2}{3}x^2 + \frac{10}{3}x - \frac{12}{3} - x^2 + 3x + \frac{14}{3}x^2 - \frac{28}{3}x$$

$$= \frac{14}{3}x^2 - \frac{2}{3}x^2 - x^2 + \frac{10}{3}x + 3x - \frac{28}{3}x - 4.$$

$$= \frac{14x^2 - 2x^2 - 3x^2}{3} + \frac{10x + 9x - 28x}{3} - 4$$

$$= \frac{9x^2}{3} - \frac{9x}{3} - 4$$

$$= \boxed{3x^2 - 3x - 4} \text{ (Ans)}$$

LAGRANGE'S INVERSE INTERPOLATION FORMULA

To evaluate x when y is given

Let $x = x_0 \quad x_1, \quad x_2 \quad \cdots \quad x_n$

$y = y_0 \quad y_1 \quad y_2 \quad \cdots \quad y_n$

Then
$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} \times x_0$$
$$+ \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} \times x_1$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_3)\cdots(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)\cdots(y_2-y_n)} \times x_2$$
$$\vdots$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_2)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)\cdots(y_n-y_{n-1})} \times x_n$$

Example ① ~~Apply~~ Lagrange's method inversely to find x when $y=15$ if

x:	5	6	9	11
y:	12	13	14	16

Sol^n Given $x_0 = 5 \quad x_1 = 6 \quad x_2 = 9 \quad x_3 = 11$

$y_0 = 12 \quad y_1 = 13 \quad y_2 = 14 \quad y_3 = 16$

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} \times x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} \times x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} \times x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} \times x_3$$

$$x = \frac{(15-13)(15-14)(15-16)}{(12-13)(12-14)(12-16)} \times 5 + \frac{(15-12)(15-14)(15-16)}{(13-12)(13-14)(13-16)} \times 6$$

$$+ \frac{(15-12)(15-13)(15-16)}{(14-12)(14-13)(14-16)} \times 9 + \frac{(15-12)(15-13)(15-14)}{(16-12)(16-13)(16-14)} \times 11$$

$$= \frac{2 \times 1 \times -1}{-1 \times -2 \times -4} \times 5 + \frac{3 \times 1 \times -1}{1 \times -1 \times -3} \times 6 + \frac{3 \times 2 \times -1}{2 \times 1 \times -2} \times 9$$

$$+ \frac{3 \times 2 \times 1}{4 \times 3 \times 2} \times 11$$

$$= \frac{5}{4} - 6 + \frac{27}{2} + \frac{11}{4}$$

$$= \frac{5 - 24 + 54 + 11}{4}$$

$$= \frac{46}{4}$$

$$= \boxed{11.5}$$

Ex-2 Apply Lagranges Formula inversely to obtain a root of the eqn $f(x)=0$, given that $f(30)=-30$, $f(34)=-13$, $f(42)=18$

Soln Given $x_0 = 30 \quad x_1 = 34 \quad x_2 = 42$

$y_0 = -30 \quad y_1 = -13 \quad y_2 = 18$

We have to find root of the eqn $f(x)=0$

i.e to find value of x when $y=0$

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} \times x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} \times x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} \times x_2$$

$$= \frac{(0+13)(0-18)}{(-30+13)(-30-18)} \times 30 + \frac{(0+30)(0-18)}{(-13+30)(-13-18)} \times 34 + \frac{(0+30)(0+13)}{(18+30)(18+13)} \times 42$$

$$= \frac{13 \times -18}{-17 \times -48} \times 30 + \frac{30 \times -18}{17 \times -31} \times 34 + \frac{30 \times 13}{48 \times 31} \times 42$$

$$= -8.6029 + 34.8387 + 11.008)$$

$$= \boxed{37.2439} \text{ Ans.}$$

NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called numerical integration.

This process when applied to function of a single variable is called QUADRATURE

The approximate value of an integral can be found by following Formula.

Newton-cotes Quadrature Formula

$$\int_{x_0}^{x_n} f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

By putting $n=1$ and neglecting all higher order differences starting from $\Delta^2 y_0$ on wards.

we get

TRAPEZOIDAL RULE

$$\boxed{\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right]}$$

By putting $n=2$ and neglecting all higher order differences starting from $\Delta^3 y_0$ on wards

we get

SIMPSON's $\frac{1}{3}$ RULE

$$\boxed{\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2})\right]}$$

Notes → * x_0 = Lower limit of the integral

* x_n = upper limit of the integral

* $\boxed{h = \frac{x_n - x_0}{n}}$

* n = Number of sub interval.

* While applying simpson's rule n must be even

* When the range is divided into 'k' equal parts then $\boxed{h = \frac{x_n - x_0}{k}}$

* when we taking 'k' no. of ordinates then $\boxed{h = \frac{x_n - x_0}{k-1}}$

Ex:- ① Using Trapezoidal Rule evaluate $\int_0^1 x^3 dx$ considering Five subinterval

Sol^n We have to take 5 subintervals.

i.e $n = 5$

$\Rightarrow h = \frac{x_n - x_0}{n} = \frac{1-0}{5} = 0.2$

x	$y = x^3$	
0	0	$\to y_0$
0.2	0.008	$\to y_1$
0.4	0.064	$\to y_2$
0.6	0.216	$\to y_3$
0.8	0.512	$\to y_4$
1	1	$\to y_5$

$\int_0^1 x^3 dx = \frac{h}{2}\left[y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)\right]$

$= \frac{0.2}{2}\left[0 + 1 + 2(0.008 + 0.064 + 0.216 + 0.512)\right]$

$= \frac{0.2}{2}\left[1 + 2(0.8)\right]$

$= 0.1\ [2.6]$

$= \boxed{0.26}$ Ans.

Ex-2 Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using (i) Trapezoidal Rule
(ii) Simpson's $\frac{1}{3}$ Rule

taking $h = 1/4$

Hence compute an approximate value of π in both cases.

Sol^n Given $h = \frac{1}{4} = 0.25$

x	$y = \frac{1}{1+x^2}$	
0	1	$\to y_0$
0.25	0.9412	$\to y_1$
0.5	0.8	$\to y_2$
0.75	0.64	$\to y_3$
1	0.5	$\to y_4$

(i) Trapezoidal Rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[y_0 + y_4 + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{0.25}{2}\left[1 + 0.5 + 2(0.9412 + 0.8 + 0.64)\right]$$

$$= \frac{0.25}{2}\left[1.5 + 2(2.3812)\right]$$

$$= \frac{0.25}{2}\left[1.5 + 4.7624\right]$$

$$= \boxed{0.7828}$$

Hence $\int_0^1 \frac{dx}{1+x^2} = 0.7828$

$\Rightarrow \left[\tan^{-1}x\right]_0^1 = 0.7828$

$\Rightarrow \tan^{-1}1 - \tan^{-1}0 = 0.7828$

$\Rightarrow \pi/4 - 0 = 0.7828$

$\Rightarrow \pi = 4 \times 0.7828$

$\Rightarrow \boxed{\pi = 3.1312}$

(ii) Simpson's $\frac{1}{3}$ Rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\left[y_0 + y_4 + 4(y_1 + y_3) + 2(y_2)\right]$$

$$= \frac{0.25}{3}\left[1 + 0.5 + 4(0.9412 + 0.64) + 2(0.8)\right]$$

$$= \frac{0.25}{3}\left[1.5 + 6.3284 + 1.6\right]$$

$$= \boxed{0.7857}$$

Hence $\int_0^1 \frac{dx}{1+x^2} = 0.7857$

$\Rightarrow \pi/4 = 0.7857 \Rightarrow \pi = 4 \times 0.7857 = \boxed{3.1428}$

Q.3 Find an approximate value of $\log_e 5$ by calculating to 4 decimal places by simpson's $\frac{1}{3}$ rule $\int_0^5 \frac{dx}{4x+5}$, dividing the range in-to 10 equal parts

Sol^n $h = \frac{x_n - x_0}{10} = \frac{5-0}{10} = \frac{1}{2} = 0.5$

x	$y = \frac{1}{4x+5}$	
0	0.2	$\to y_0$
0.5	0.1429	$\to y_1$
1	0.1111	$\to y_2$
1.5	0.0909	$\to y_3$
2	0.0769	$\to y_4$
2.5	0.0666	$\to y_5$
3	0.0588	$\to y_6$
3.5	0.0526	$\to y_7$
4	0.0476	$\to y_8$
4.5	0.0435	$\to y_9$
5	0.04	$\to y_{10}$

$$\int_0^5 \frac{dx}{4x+5} = \frac{h}{3}\left[y_0 + y_{10} + 4(y_1+y_3+y_5+y_7+y_9) + 2(y_2+y_4+y_6+y_8)\right]$$

$$= \frac{0.5}{3}\left[0.2+0.04 + 4(0.1429+0.0909+0.0666+0.0526+0.0435) + 2(0.1111+0.0769+0.0588+0.0476)\right]$$

$$= \frac{0.5}{3}\left[0.24 + 1.5860 + 0.5888\right]$$

$$= \boxed{0.4023}$$

$\therefore \int_0^5 \frac{dx}{4x+5} = 0.4023 \implies \frac{1}{4}\left[\log_e(4x+5)\right]_0^5 = 0.4023$

$\Rightarrow \frac{1}{4}\left[\log_e 25 - \log_e 5\right] = 0.4023$

$\Rightarrow \log_e \frac{25}{5} = 4 \times 0.4023 \Rightarrow \boxed{\log_e 5 = 1.6092}$

Ex-4 Use Simpson's $\frac{1}{3}$ Rule find $\int_0^{0.6} e^{-x^2}\,dx$ by taking seven ordinates.

Solution We have to take 7 ordinates

that means $n=6$

$$h = \frac{x_n - x_0}{n} = \frac{0.6-0}{6} = 0.1$$

x	$y = e^{-x^2}$	
0	1	$\to y_0$
0.1	0.99	$\to y_1$
0.2	0.9608	$\to y_2$
0.3	0.9139	$\to y_3$
0.4	0.8521	$\to y_4$
0.5	0.7788	$\to y_5$
0.6	0.6977	$\to y_6$

$$\int_0^{0.6} e^{-x^2}\,dx = \frac{h}{3}\left[y_0 + y_6 + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{0.1}{3}\left[1 + 0.6977 + 4(0.99 + 0.9139 + 0.7788) + 2(0.9608 + 0.8521)\right]$$

$$= \frac{0.1}{3}\left[1.6977 + 10.7308 + 3.6258\right]$$

$$= \boxed{0.5351} \text{ Ans.}$$